Delay Recovery from a Non-linear Polynomial-Response System

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Abstract

In this paper, we present a technique to determine a signal delay as it passes through a polynomial response non-linear system. Where the nature of the non-linearity is not known, the technique is capable of finding the order of the response, and in the case of multi-ordered responses, the delay of each order. This technique also works for linear systems.

For non-linear systems, the usual characterisation approach is to use minor perturbations of a linear model (Volterra), to use stochastic inputs and model the system statistically (Wiener), or to use blind system identification techniques. Here we use digital signatures, embedded on a carrier, to measure system delay.

1. Introduction

Conventional approaches to the identification of an unknown system for the purpose of mathematically modelling it have assumed (if at all possible) that the model be linear. This greatly simplifies the establishment of system limits and a large array of experimental techniques is available for determining these model parameters [3,12].

For non-linear systems, delay measurement is problematic [11]. The system demonstrated here uses a carrier with an embedded signature to characterise the delay of the system.

The use of pseudo-noise (PN) sequences for the identification of a system has been well established [7].

The process is generally two-pronged, a time-domain analysis involving correlational methods to derive an impulse response function, and a frequency-domain approach yielding a transfer function.

The excellent auto- and cross-correlation property of M-sequences is used to advantage here [9,17].

2. Linear System Measurements

Identification of linear system parameters is a field with well-established protocols and methods [3,11,12]. These almost invariably require the assumption that a signal at time t_1 has no effect on the response at time t_2 (shift-invariance). The principal result of this is that an impulse response or transfer function can be defined independently of the inputs used to obtain them.

3. Non-Linear Systems

Non-linear systems are essentially those that break the superposition principle.

Volterra [13] first described a method for identifying the parameters of a non-linear system by approximating it to a linear system for small stimulus. Pinter et al. [6] gives an excellent introductory overview of non-linear methods applied to biological systems.

In practice, one would first obtain the first order Volterra kernel approximation, using very small stimuli, so that higher order terms might approach zero. Subsequently, to get the 2nd order term, one would compare the response of two small stimuli to the response of their sum.

It is this ‘smallness’ requirement (and this definition is application dependent) that significantly undermines the effectiveness of the Volterra approach. Measurement of small differences is inherently noisy and prone to error.

The algorithm presented in this paper seeks to improve the reliability and generality of system characterisation.

4. Properties of PN sequences

There are a number of ways to extract a digital signature from a system output after the signature is added to the input signal. The most common of these is to cross-correlate the output signal against the phase-aligned input signature (or to perform the equivalent function in the frequency domain).

A desirable feature of the signature is that its periodic auto-correlation is two-valued, returning one (large) value for in-phase and the other (small) value for all out-of-phase components. Such sequences are called pseudo-noise (PN) sequences.

There are many sequences with this singular characteristic [14,18]. Among them are the M-sequences, twin-prime, Hall, and Legendre sequences [15], approximations such as Gold [16] and Kasami sequences [14], as well as strategies for searching exhaustively for these sequences of arbitrary length [8].

The strategy in this paper is to use sequences with such analytically determinable properties to demonstrate an ability to extract phase-sensitive information.

The digital signatures can be either binary or multi-valued.

While strong signature recovery is obviously a requirement, it is also necessary to consider the effect of the addition of the signature on the system output. Minimum disturbance of the system (signal) statistics is achieved by ensuring that the signature appears noise-like, with Gaussian or uniform distribution, with zero mean and a flat spectral distribution.

This requirement narrows the choice of possible candidate sequences, and in this paper, the use of M-sequences (maximal-length recursive sequences – see MacWilliams and Sloane [4]) and Legendre sequences [5] is discussed.

4.1 The Binary M-sequence

Zierler [9] first introduced the concept of M-sequences and showed their advantages as a binary sequence. They have since been used in many areas for signal phase recovery in noisy environments in such diverse areas as communication [4], concert hall acoustics [5] and digital image signature recovery [1,2].

For our needs, they share the following properties:

- **Unique auto-correlation.** A single auto-correlation peak, of magnitude $2^n - 1$, occurring at zero phase and -1 at all other phases.
- **Indistinguishable from Gaussian noise when added to the data.** The statistics of binary M-sequences exactly match those of ideal uniform statistics for 1-bit additive noise. The run-lengths are Gaussian distributed.
- **Easy to generate using shift registers.** The simple shift registers required are cheap and easy to model and build.
- **Balanced.** Since the number of 1's exceeds the number of 0's by just 1, there is a negligible bias added to the data, $1/(2^n-1)$.
- **Spectrally Uniform**. Because of their unique auto-correlation, M-sequences, when transformed to frequency domain, uniformly cover the spectral range determined by their length. As shown in figure 2, if the M-sequence is sampled at a rate higher than their highest switching rate, the (larger) spectrum envelope will resemble that of a single triangular pulse. The envelope width corresponds inversely to that of one full-length M-sequence cycle with a $(\sin(x)/x)^2$ drop-off.
- **Predictable Cross-correlations.** Different M-sequences can be used to encode more than one signature on the same data (This might be done at different rates, to determine different scale effects, for example). Interactions between different M-sequences are well understood [14,18].

4.2 Extension to Multi-valued M-Sequences

Although binary M-sequences are traditionally used for signature purposes, the concept extends in a straightforward manner to higher-valued finite fields [5].

All the properties listed above for binary M-sequences extend to the multi-valued case with the following additions:

- Gaussian run-length, uniform amplitude distribution.
- Shift registers can be used for any power-of-prime M-sequence (of length p^n-1).
- Balance can be maintained, given an appropriate symbol allocation (see below).

The M-sequence owes its properties to it being a residue system. A residue system modulo a prime p forms a finite number field of order p. Excluding 0, one may arrange the symbols in this field to be powers of α with all α^n being primitive roots of unity:

$$\alpha^0, \alpha^1, \alpha^2, \alpha^{p-1} = 1 \quad (1)$$

These primitive roots may be mapped to real numbers, as done in Barker [7], to define a pseudo-noise sequence X_I,

$$X(\alpha^0), X(\alpha^1), X(\alpha^2), \dots, X(\alpha^{p-1}). \quad (2)$$

Further, it is possible to define a primitive polynomial over which the algebraic field laws all apply, modulo p.

MacWilliams and Sloane [4] show that the number of distinct irreducible primitive polynomials over a field of order p^m is

$$\phi(p^m-1)/m \quad (3)$$

where p is prime, m is a positive integer and $\phi()$ is the Euler Totient function. $\phi(i)$ is defined as the number of integers from 1 to i-1 which have no common divisor with i.

Each of these irreducible polynomials defines a field where the powers of α will cover every value in the field cyclically, modulo p, exactly once, before repeating.

Many sequences of power-of-prime length possess low autocorrelation side-lobe behaviour [8].

4.3 Angle Multiplication

When a roots-of-unity sequence embedded on a sinusoidal carrier is subjected to a non-linear transformation, the output contains multiple components, at different harmonics of the input. The sequences embedded in these harmonic signals are powers of the input sequence.

Consider a ternary sequence: with alphabet 1, α, α^2, all being roots of unity and thus forming a finite (Galois) field, we may express these values as angles on the unit circle

$$S_i = \exp(j\omega t + \psi(i)) \tag{4}$$

where $\psi(t)$ is a time-varying phase offset. For a GF(3) signal, it will take the values of $2n\pi/3$, for n = 0, 1, 2 corresponding to the {1, α and α^2} terms. For the n=3 term, α^3 modulo 3 wraps to α^0, i.e. n=0.

If the system has a 2nd order effect, there will be a component:

$$\begin{aligned} S_i^2 &= \exp^2(j\omega t + \psi(i)) \\ &= \exp(2j\omega t + \psi'(t)) \end{aligned} \tag{5}$$

with $\psi'(t) = 2\psi(t)$, so that all its symbols (angles) are doubled yielding an alphabet {1, α^2, α^4}. For a ternary signal α^4 (modulo 3) = α, so there has been a conjugation of part of this sequence alphabet to {1, α^2, α}.

Similarly, one might imagine a 3rd order effect for this ternary signal producing the alphabet 1, α^3, α^6. Unfortunately, these all map onto 1. Turning this to advantage, it is possible to assign a field such that individual order effects (in this case, 3rd order) can be filtered out.

Costas [10] used just this technique to recover the carrier from a phase modulated signal by quantising the phase modulation at the sender into M discrete levels and the receiver multiplying the received signal by M, effectively removing the phase modulation. This process is also utilised in Quadrature Amplitude Modulation (QAM).

To maintain a one-to-one mapping of symbols, over an nth order transformation, it is necessary to use a power-of-prime field, p, with p greater than n which contains the least common multiple between all values of 1 to n.

5. Phase Modulated Input Signals

Figure 1 shows a non-linear polynomial-response system whose inputs are (for simplicity) sinusoidal. If we phase-modulate this input signal with a pseudo-noise (PN) sequence as described above, the system response will modify the signal and the embedded signature.

In this way, the procedure of the Volterra / Wiener [11] approach can be emulated using pseudo-noise perturbations instead of Gaussian noise. The auto-correlation property of the signature allows a large ensemble to be measured permitting greater noise immunity.

Judicious use of digital filters makes it possible to isolate higher frequency harmonics individually. Greater noise immunity is possible through the use of long sequences.

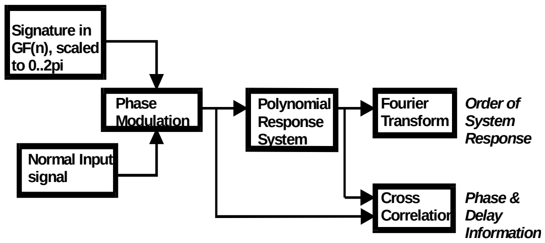

Figure 1. A block diagram of the proposed system measurement

The symbol conjugation can also be used to advantage. A periodic cross-correlation of input and output will only correlate against the S component (eq 4) of the response. The higher order S^i have each suffered a symbol transformation, thus the cross-correlation will not respond to them directly.

In order to isolate a particular order of non-linear system response, the correlation template is distorted appropriately to that order (by multiplying *all* angles).

6. An Example

By way of example, we will simulate a system to observe up to 4th order effects (including interaction between orders).

To observe 4th order effects, the field needs to be large enough to encompass the least common multiple for 4th, 3rd, and 2nd order effects (2x3x4=12).

A smaller field of symbols will improve reliability in angle resolution, as there are fewer symbols around the unit circle, thus allowing for greater error tolerance in the measured phase shifts.

Figure 2. shows a screen shot of the test system developed on a PC using Labview 3.1.1.
The horizontal axis represents time delay in all but the ‘FFT’ and ‘Mod/Polymod’ windows. The top window, ‘Mseq Array’, shows one iteration of the GF(31) signature. This signal is applied to the phase of the sine carrier with period 11.7 samples, and sampled 39 times for each transition of the signature (‘Super sample’) in the second window (‘Mod Carrier’). The second window shows about 3 cycles of the phase modulation. The third window shows the input signal transformed by a polynomial whose coefficients, a, are listed to the centre left (with a cubic term shown as selected).

The fourth window shows the Fourier Amplitude of the output. This displays frequency tripling (in this case). The lowest window shows the measured delay estimation (delay was 100 samples, as shown). The 2D display ‘Mod/PolyMod’ in the lower-left is a crude rule-of-thumb test graph showing the sampling distribution of the modulated waveform.

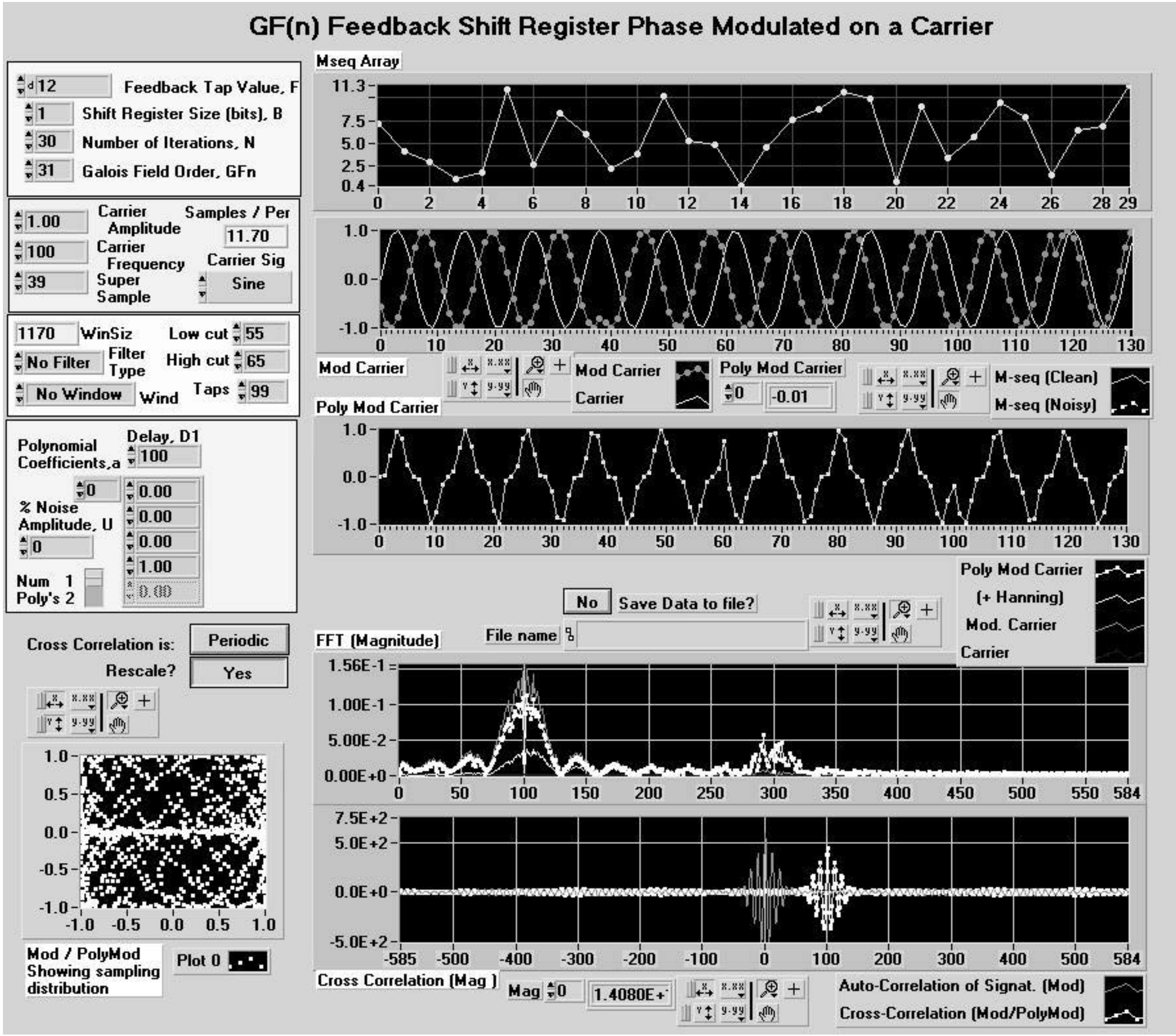

Figure 2. Screen shot of test system.

A sparse display on this graph would indicate that few different values of I(t) are being sampled and that signal aliasing is likely.

Note that the second and third windows have been scaled to display only the first 130 of the 1170 samples (=39*31) for clarity. Note also the signal amplitudes, especially in the ‘cross-correlation’ window.

The triangular envelope correlation output is clearly visible, as is the $(\sin x / x)^2$ envelope in the Fourier spectrum.

If a longer sequence is used, the correlation envelope narrows (and its Fourier envelope expands). This represents the usual tradeoff between frequency and delay resolution. Note that the high frequency component of the correlation output corresponds to that of the carrier.

Figure 3 shows the same result, but with of 200% noise added. Although the FFT appears now to be completely swamped by the noise, the delay value is still readily recoverable. The delay recovery can be improved further by using longer sequences.

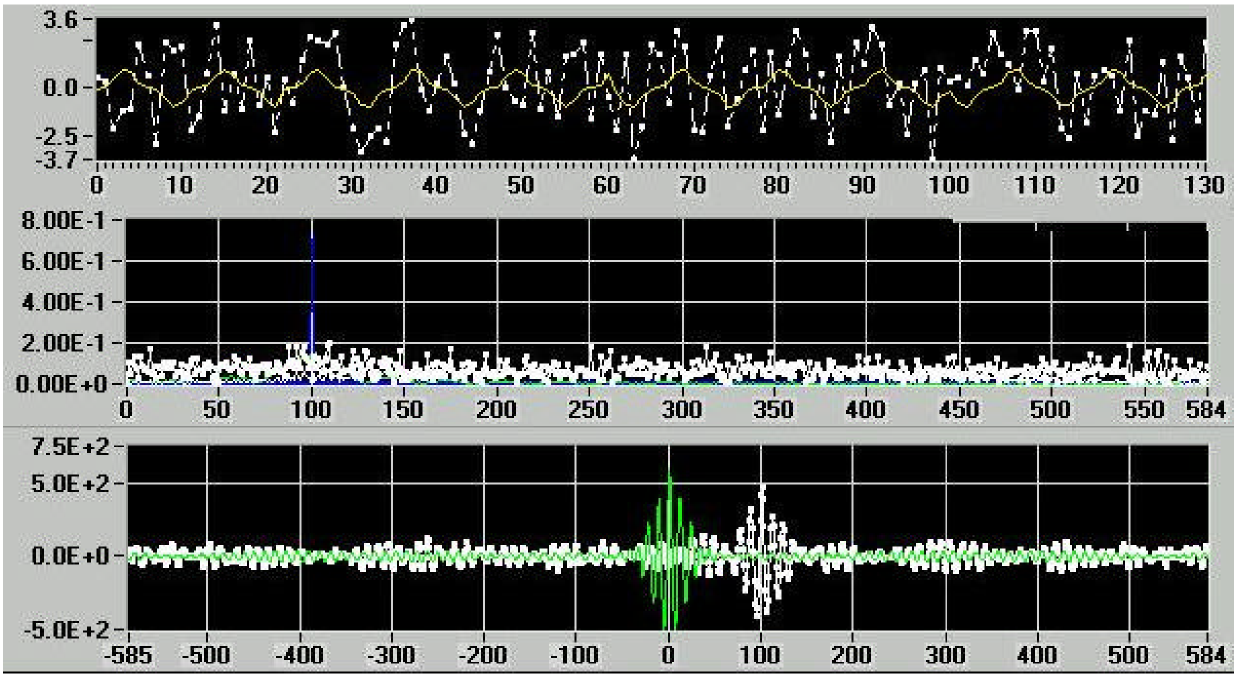

Figure 3. The 'Mod Carrier', 'FFT', and 'Cross-correlation' windows of the test system of Figure 2 with the same legend and the same settings, but with 200% uniform noise added.

7. Conclusion

A novel approach has been demonstrated to determine the time delay in a system with a non-linear response characterised by a polynomial of unknown order. The input signal is phase shift encoded with a suitable sequence. This enables the separation of different orders in the non-linear response and thus independent delay measurement for each order of response.

8. References

1. A.Z. Tirkel, T.E. Hall, C.F. Osborne., *Steganography-Applications of Coding Theory*. IEEE Information Theory Workshop, Svalbard 1997, p. 59-60.
2. A.Z. Tirkel, G.A. Rankin, R.M.van Schyndel, W.J. Ho, N.R.A. Mee, C.F. Osborne. *Electronic Water Mark*. DICTA-93, Sydney, December 1993. p. 666-672.
3. J. Juang, *Applied System Identification*, 1994, Prentice-Hall International, London.
4. F. J. MacWilliams, N. J. A. Sloane, *The Theory of Error-Correcting Codes*, 1978, North-Holland.
5. M. R. Schroeder, *Number Theory in Science and Communications*, 3rd Ed., 1997, Springer-Verlag.
6. R. B. Pinter, B. Nabet (eds), *Non-linear Vision: Determination of Neural Receptive Fields, Function, and Networks*, 1992, CRC Press
7. H. A. Barker, R. W. Davy, *Measurement of Second Volterra Kernels using Pseudo Random Signals*, Int. J. Control., 1978, vol 27, p. 277-291.
8. S. Mertens, *Exhaustive Search for Low Autocorrelation Binary Sequences*, Letter to the Editor, J. Phys. A: Math. Gen. 29(1996), p. L473-L481.
9. N. Zierler, *Linear Recurring Sequences*, J. Soc. Ind. Math., 1959, Vol 7, p. 31-48.
10. J.P. Costas, *Synchronous Communications*, Proc. IRE, vol 44, p. 1713-1718, December 1956.
11. J. Bendat, *Non-Linear System Analysis and Identification from Random Data*, 1990, John Wiley & Sons.
12. T. Söderström, P. Stoica, *System Identification*, Prentice-Hall International, London, 1989.
13. V. Volterra, *Theory of Functional and Integral and Integro-Differential Equations*, 1930, Blackie, London.
14. D.V. Sarwate, M.B Pursley, *Crosscorrelation Properties of Pseudorandom and Related Sequences* Proc. IEEE vol.68, no. 5. May 1980, p. 593-619.
15. S.Kitabayashi, T.Ozawa, M.Hata, *Property of the Legendre Subsequence*, Communication on the Move. ICCS/ISITA '92. Singapore. vol 3. p. 1224-1228.
16. R. Gold, *Optimal Binary Sequences for Spread Spectrum Multiplexing*, IEEE Trans. on Information Theory, vol IT-13., no 4, October 1967, p. 619-621.
17. S.W. Golomb, *Shift Register Sequences*, Holden-Days Inc., San Francisco Ca., 1967.
18. A.Z. Tirkel, C.F. Osborne, T.E. Hall, *Effects of Bias and Characteristic Phase on the Cross-Correlation of M-sequences*, IEE Proceedings, Communications vol 144, No. 4, August 1997