Comments on "Design of Decentralized Control for Symmetrically Interconnected Systems"

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Abstract—In a recent paper [Kashlan and Geneidy (1996). Design of decentralized control for symmetrically interconnected systems, Automatica 32(3), 475–476] a necessary and sufficient condition is reported for global eigenspectrum assignment by a decentralized state feedback control. We argue that the condition does not hold. © 1998 Elsevier Science Ltd. All rights reserved.

In a recent paper (Kashlan and Geneidy, 1996), the authors introduce a lemma for global eigenspectrum assignment via decentralized state feedback control law. In this note we note that this lemma is incorrect. We also highlight other flaws in the paper.

\[(A_i - \lambda_i I)v_{i1} + A_{i1}v_{1i} + \ldots + A_{iN}{v}_{Ni} + B_{1i}w_{1i} + \ldots + B_{Ni}w_{Ni}\]

\[(A_i - \lambda_i I)v_{i2} + A_{i1}v_{1i} + \ldots + A_{iN}{v}_{Ni} + B_{2i}w_{1i} + \ldots + B_{Ni}w_{Ni}\]

\[\vdots\]

\[(A_i - \lambda_i I)v_{iN} + A_{i1}v_{1i} + \ldots + A_{iN}{v}_{Ni} + B_{1i}w_{1i} + \ldots + B_{Ni}w_{Ni}\]

\[= 0. \quad (c2)\]

This statement does not help to prove the lemma since the fact that the RHS = 0 means exactly the same as \((A + \Lambda)\) and \(\Lambda\) have no common eigenvalues, a statement which needs to be proven.

Remark 2. Equation (12) is obtained on the assumption that the open and closed-loop eigenvalues are not the same. Therefore, it cannot be used as the basis for proving the assumption it is based on.

Remark 3. The RHS of equation (15) must be equated to zero.

Remark 4. From equation (14) the following set of equations is obtained for \(i = 1, 2, \ldots, N\).

\[QV_i + RW_i = 0. \quad (c3)\]

where

\[Q = [A + \Lambda - \lambda_i I]; \quad R = [B + B];\]

\[V_i = [v_{i1} \; v_{i2} \; \ldots \; v_{iN}]^T;\]

\[W_i = KV_i = [w_{i1} \; w_{i2} \; \ldots \; w_{iN}]^T\]

and

\[K = \text{diag}(K_1, K_2, \ldots, K_N).\]

Although the authors have not made it clear as to how one would solve for \(V_i\) in equation (2c), one might postulate the following two possibilities:

(i) by freely selecting the elements of \(W_r; r = 1, 2, \ldots, N\) sequentially starting from \(i = 1\) and proceeding to \(i = N\). If this is the case then, for say \(r = 1\), one is left with a set of \(n\) simultaneous equations with \(n + m - m\), unknowns. This set of equations is inconsistent with the set obtained from, say \(r = 2\). Hence, it is not possible to solve for \(V_i\). In addition, any solution obtained by solving each equation in equation (2c) individually cannot be guaranteed to satisfy the rest of the equations.

(ii) by freely selecting the elements of all \(W_r; r = 1, 2, \ldots, N\) simultaneously to determine \(W\). In this case the problem becomes that of solving a set of \(n\) linear simultaneous equations with an unknown for which a solution exists if

\[\text{rank}[Q : -RW] = \text{rank}[Q] = n. \quad (c4)\]

As the rank of \(Q\), under the assumption (c1), is full, obviously any choice of \(W\) will satisfy the rank condition (c4). But this is not the full story. Rather the key problem that needs to be resolved is how one might choose \(W\) so that upon subsequent evaluation of \(V_i\) and \(K\), equation (18) is satisfied. There is no

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guidance in the paper how this can be achieved in a systematic fashion.

Note that for case (ii) the development from equation (9) to equation (16) in the paper is redundant. In addition, the procedure is not executed at the subsystems level in the sense that $n$ simultaneous equations have to be solved $N$ times instead of solving $n$, set of equations $N$ times.

Remark 5. The authors state that one might exploit the additional parameters $(mm - \sum_{i=1}^{n} m_i, m_i)$ to choose $V$ to satisfy equation (20). This may not be possible at all because, $V$ is in fact calculated and not chosen, as stated by the authors (see Remark 4 above). The only chosen parameter is $W$. A simple example by the authors could have answered all of the concerns raised above.

In the following we demonstrate by a counterexample that the lemma of the paper in fact does not hold.

Counterexample. Consider the following counterexample:

\[ X_1 = X_1 + 2X_2 + u_1 + 2u_2, \]
\[ X_2 = 2X_1 + X_2 + u_1 + u_2. \]

Thus we have

\[ A + \tilde{A} = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \]

The open-loop eigenvalues of this system are $-1$ and $3$ and the two subsystems are controllable.

Let a decentralized controller of the form $K = \text{diag}(k_1, k_2)$ be used. Then the closed-loop system is not held as

\[ \begin{pmatrix} k_1 + 1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} 1 + k_1 & 2 + zk_2 \\ 2 + zk_1 & 1 + k_2 \end{pmatrix} \]

If we set $z = \beta = \pm \frac{1}{\sqrt{2}}$, then the closed-loop eigenvalues are determined as

\[ \lambda = \frac{2 + k_1 + k_2 + \sqrt{(k_1 + 4\beta)^2 + (k_2 + 4\beta)^2}}{2} \]

From this it is clear that for any specified closed-loop complex pole $\lambda \in \sigma(A + \tilde{A})$, the decentralized controller $K$ has no solution in the real field.

Authors reply

In our paper (Kashlaf and Geneidy, 1996) we presented an algorithmic solution to the problem of designing the decentralized controllers that achieve a prescribed set of eigenvalues to the "global" closed system which is composed of symmetrically interconnected subsystems. This problem assumed central importance in linear control theory.

We present here the reply to the comments of Hu and Aldeen (1997).

(i) The starting point in our design procedure is a well-known equation in matrix algebra known as Sylvester equation (Bhathtacharya, 1982), namely equation (10). Full details of the general algebraic solution for such an equation are given earlier by Gantmacher (1959), and the necessary and sufficient conditions for its solution are found in any classical text book (Barnett and Storey, 1970). This equation finds wide applications in control system design as it is closely related to the problem of eigenspectrum assignment from about two decades (Bhathtacharya, 1982) up to present, we take Chiang and Throp (1993), and Duan (1994, 1996) as examples. Our lemma is a consequence of Sylvester condition. The authors did not prove in their comments (Remarks 1 and 2) that the necessity nor the sufficiency of our lemma holds not for the following reasons:

(i) The necessary condition states that if $K$ exists then $\Omega \cap \sigma(A + \tilde{A}) = \emptyset$, in the authors discussion they did not prove that the condition is not true.

(ii) The sufficient condition states that if $\Omega \cap \sigma(A + \tilde{A}) = \emptyset$, then $K$ exists. This means that $[(A + \tilde{A}) - \lambda I]^{-1}$ exists, $V_i \neq 0$; hence $K$ is nontrivial as stated two lines after equation (12). As for Remark 2 it coincides with our necessary and sufficient conditions, we cannot understand clearly why it is false as claimed by the authors.

(2) As for Remark 3 we agree that the right-hand side of equation (15) equals zero, as it is a restatement of equation (14). In fact this is typographical error.

(3) The last equation in equation (c2) has incorrect notation.

(4) The main question posed in Remark 4 is whether the system of equations (c2) has a solution and how this solution can be obtained.

As stated in our abstract the solution procedure is executed at the subsystem level. The previous question can be answered by using the additional available degrees of freedom $(nm - \sum_{i=1}^{n} m_i, m_i)$.

We note here that the authors do not pursue our work for the serious errors in their comments, specifically mixing matrices and vectors as shown in the following.

For (c2) to coincide with equation (14), the notation must be adjusted and the condition in the summation in equation (14), namely $q \neq p$, must be taken into consideration. Also to deal with equation (c3), the correct matrices and vectors must be as follows:

\[ Q = [A + \tilde{A} - \lambda I] \in \mathbb{R}^{m \times m}, \quad V = [V_1, V_2, \ldots, V_n] \in \mathbb{R}^{m \times m}, \]
\[ W = [W_1, W_2, \ldots, W_m] \in \mathbb{R}^{n \times n}, \]

\[ V_i = \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ \vdots \\ v_{i,n} \end{pmatrix}, \quad V \in \mathbb{R}^{m \times m}, \quad v_{i,0} \in \mathbb{R} \]

and

\[ W_i = \begin{pmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,n} \end{pmatrix}, \quad W \in \mathbb{R}^{n \times n}, \quad w_{i,0} \in \mathbb{R} \]

Note here the differences between these and those indicated in the comment.

Possibility (i) in the comments in selecting the free parameters contained in equation (c3) is not at all applicable in our proposal, but possibility (ii) is exactly our procedure with the exception that selecting $\omega_0$ is executed repeatedly at the subsystem level i.e. using equation (15) repeatedly yielding subsolutions. Later, subsolutions are aggregated to form the general parametric solutions (18).

In order to get a unique solution for the eigenvectors $V$ corresponding to some choice of the parameters $W$ condition (c4) should be rank $[Q: -RW] = \text{rank}[Q] = n$.

Here we must confirm that rank $[Q: -RW] = n$ if and only if $\lambda$ and $(A + \tilde{A})$ have no common eigenvalue, hence it contradicts their claim but proves our lemma at once.

The only guarantee the authors are looking for to satisfy condition (c4), namely rank $(A + \tilde{A} - \lambda I) = n$, is that $\lambda \notin \sigma(A + \tilde{A})$, which is the conclusion of our lemma.

It is not difficult to show that equation (12) leads to

\[ \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix} = \\
-[(A + \tilde{A} - \lambda I)^{-1}(B + B), \ldots, (A + \tilde{A} - \lambda I)^{-1}(B + B)] \]

\[ \times \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ O \\ \vdots \\ O \\ \vdots \\ W_m \end{bmatrix} \]
However, nothing guarantees that the resulting eigenvectors matrix is orthogonal as equation (20), nor the matrix \( W(\Theta)V^{-1}(\Theta) \) takes the special "decentralized" structure 
\[ K = \text{diag}(k_1, \ldots, k_n) \] 
from the first instant.

Our procedure for matching left–right-hand sides of equation (21) and at the same time generating orthogonal eigenvectors as equation (20) is an "open loop" in the sense that if an unaccept-
able results due, for instance, non-orthogonal eigenvectors or non-zero off diagonal blocks for the decentralized feedback matrix \( K \), other selection for the free parameters to improve the undesirable results is possible. The feedback gains are non-
unique, this may be attributed to the fact that the free parameters are selected in open loop manner. The algorithm is based on the gradient search technique that terminates when a limiting value \( e = |V - V^{-1}| \) approaches a small prespecified value.

At last two answer the frequently asked question on how to select \( \omega_i \)'s we offer the systematic procedure for solving the example given in equation (c5).

**Step 1:** Test the controllability of the pair \((A + \tilde{A})\) and \((B + \tilde{B})\).

**Step 2:** Equation (12) may be rewritten as
\[
\begin{bmatrix}
\nu_1 \\
\nu_2
\end{bmatrix} = - \begin{bmatrix}
(1 - \lambda) & 2 \\
2 & (1 - \lambda)
\end{bmatrix}^{-1} \begin{bmatrix}
1 & 0.707 \\
0.707 & 1
\end{bmatrix} \begin{bmatrix}
\omega_1 \\
\omega_2
\end{bmatrix},
\]

(\text{**})

where \( \lambda_i, i = 1, 2 \) are the desired eigenvalues and \( \omega_{1i} = k_1 \nu_{1i} \) and \( \omega_{2i} = k_2 \nu_{2i} \). Note that for any choice for \( \omega_{1i} \) and \( \omega_{2i} \) one gets the corresponding eigenvector \( \nu_{*i} \).

Also none of the \( \lambda_i \)'s (the closed loop eigenvalues) is \(-1\) or \(3\) (the open-loop eigenvalues).

**Step 3:** for \( i = 1, 2 \) one repeats equation (**) to get
\[
V_1 = \begin{bmatrix}
\nu_{11} \\
\nu_{12}
\end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix}
\nu_{21} \\
\nu_{22}
\end{bmatrix}.
\]

\[ V = [V_1, V_2] \quad \text{and} \quad W = \begin{bmatrix}
\omega_{11} & \omega_{21} \\
\omega_{12} & \omega_{22}
\end{bmatrix}.
\]

Hence, \( K = WV^{-1} \).

**Step 4:** Test
\[
\begin{bmatrix}
1 & 0.707 \quad 1 \\
0.707 & 1
\end{bmatrix} \begin{bmatrix}
\omega_{11} & \omega_{21} \\
\omega_{12} & \omega_{22}
\end{bmatrix}
\]

it must be 2.

**References**


