# A Note on an Analytic Model for Slow Start in TCP

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***Abstract*—Sikdar et al [1] has provided a formula to model the window increase pattern given the mechanics of delayed acknowledgements. In this paper, we show that their sum formula significantly underestimates the iterated sum for rounds greater than seven. The approximation error grows exponentially. We also derive a more accurate sum formula whose approximation error is linearly bounded. We show that with use of the incorrect sum formula, cases arise where it is impossible to time out as the initial loss indication in slow start, which can have significant impact on TCP latency, assuming a correlated loss model. We show that our more accurate sum formula greatly reduces the number of cases showing this behaviour, further improving upon the accuracy of the model.**

***Index Terms*—Modeling, Transport protocols.**

## I. Background

Analytic models are very useful to gain an in-depth understanding of the fundamental behaviour of TCP protocols. Recently, some efforts have been made to build analytic models for TCP performance using round-trip time and packet loss probability [1][2][3]. In [1], comprehensive analytic models are provided for TCP behaviour. However, their model for packet sums in slow start has little theoretical background, and no indication of how the formula was obtained. Their analytic sum formula is also provided to calculate the total number of packets transmitted in the first $k$ rounds in the slow start phase of TCP. We will show that their sum formula significantly underestimates the iterated sum for $k > 7$. We also derive a more accurate sum formula.

In this note, the same notations from [1] are adopted. In [1], Sikdar et al have presented a formula to model the window increase pattern, given the mechanics of delayed acknowledgements. This is presented as

$$w(n) = \lfloor 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}} \rfloor, \qquad (1)$$

where $\lfloor \ \rfloor$ is the floor function. Equation (1) is used to determine the number of packets transmitted in the $n^{\text{th}}$ round. This forms the basis of further models to eventually determine the number of rounds required, and hence the transfer latency, for a slow start period ending in either packet loss or reaching the slow start threshhold, *ssthresh*. A formula to determine the total number of packets transmitted in $k$ rounds is presented as

$$\begin{aligned} pkt(k) &= \sum_{n=1}^{k} w(n) \\ &= \lfloor 2^{\frac{k+1}{2}} + 3(2)^{\frac{4k-3}{8}} - 2 - 3/\sqrt{2} \rfloor. \end{aligned} \qquad (2)$$

TABLE I

Comparison of calculated sum, formula sum, and proposed sum of packets sent in $k$ rounds.

| $k$ | Calculated sum | $pkt(k)$ | $pkt_{new}(k)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 3 |
| 3 | 6 | 6 | 6 |
| 4 | 10 | 10 | 10 |
| 5 | 16 | 16 | 16 |
| 6 | 25 | 25 | 25 |
| 7 | 38 | 38 | 38 |
| 8 | 57 | **55** | 57 |
| 9 | 84 | 80 | 84 |
| 10 | 122 | 115 | 122 |
| 11 | 176 | 164 | 176 |
| 12 | 253 | 234 | 253 |
| 13 | 362 | 333 | 362 |
| 14 | 516 | 473 | 516 |
| 15 | 734 | 670 | 734 |
| 16 | 1043 | 950 | **1042** |

Due to the presence of the floor function, rounding error is present for each summand, excluding the possibility of presenting the sum as a simple geometric series result. Table I shows a sample of comparisons between $pkt(k)$ and the calculated sum of $w(n)$. We show that this formula does not correctly sum up the number of packets sent per round once $k$ increases beyond seven rounds. The sum can also be shown to significantly underestimate the calculated sum for $k > 7$.

The formula for $pkt(k)$ is then used to calculate other parameters such as the number of rounds needed to transmit $N$ packets. It is also integral in determining the number of lost

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packets in conjunction with a correlated loss model adopted from [3], which determines whether the TCP flow being modeled will enter Fast Retransmit or time out [4]. We will show in Section IV that by introducing a crude approximation to the sum of packets sent, we get the effect of eliminating the possibility of time out in TCP as the initial loss indication, when the first lost packet's sequence number, $i$, is greater than 55. The error is hence propagated to other formulae in their paper.

## II. A New Analytic Sum Formula

In this section, a new and more accurate sum formula is derived as follows. Ignoring the floor function in (1) we obtain

$$S_n = 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}}. \tag{3}$$

The sum of (1) could then be approximated through a simple geometric series, and is given by

$$\begin{aligned} pkt_{gs}(k) &= \sum_{n=1}^{k} S_n \\ &= \left(1+\sqrt{2}/2\right)\left(1-\sqrt{2}^{k}\right)\left(1-\sqrt{2}\right) \\ &= 2\left(\sqrt{2}\right)^{k} + 3\left(\sqrt{2}\right)^{k-1} - 2 - 3\sqrt{2}/2 \\ &= 2^{\frac{k+2}{2}} + 3\,(2)^{\frac{k-1}{2}} - 2 - 3\sqrt{2}/2. \end{aligned} \tag{4}$$

By (1) and (3), we know that the exists an error $\delta_n = S_n - w(n)$ where $0 \le \delta_n < 1$. This quantity is derived, as shown by

$$\begin{aligned} \delta_n &= \begin{cases} 2^{\frac{n-1}{2}} - \lfloor 2^{\frac{n-1}{2}} \rfloor & \text{for even } n \\ 2^{\frac{n-2}{2}} + \lfloor 2^{\frac{n-2}{2}} \rfloor & \text{for odd } n \end{cases} \\ &= 2^{\lfloor \frac{n}{2} \rfloor} 2^{-\frac{1}{2}} - \lfloor 2^{\lfloor \frac{n}{2} \rfloor} 2^{-\frac{1}{2}} \rfloor \\ &= << 2^{\lfloor \frac{n}{2} \rfloor} 2^{-\frac{1}{2}} >>, \end{aligned} \tag{5}$$

where $<< x >>$ indicates the fractional part of $x$.

We recall that $\alpha n$ is uniformly distributed modulo one if $\alpha$ is irrational and $n$ runs through all positive integers [5]. Standard references on irrational numbers such as [5], [6] do not guarantee that $\alpha 2^n$ is definitely uniformly distributed modulo one for irrational $\alpha$ as used in our applications. Heuristically, a natural choice of approximation is to proceed as though our instances of $\alpha 2^n$ are actually uniformly distributed modulo one.

The benefit is that our approximation is tractable and can lead to a much more accurate result than that provided by [1]. By treating $\delta_n$ heuristically as though it was uniformly distributed, we can further improve the accuracy of (4) by subtracting the average of 1/2 (for a uniform distribution between 0 and 1) per round. We also subtract a small bias 0.035 to ensure that the crucial early rounds are correct where the number of error samples is too small to yield an average of 1/2. This leads to our new sum formula

$$pkt_{new}(k) = \left\lfloor 2^{\frac{k+2}{2}} + 3\,(2)^{\frac{k-1}{2}} - \frac{k}{2} - 2 - 3\sqrt{2}/2 - \frac{7}{200} \right\rfloor. \tag{6}$$

Equation (6) can be shown to be correct for $k$ less than 16, see Table I, where boldface indicates the first instance of error in the corresponding formulae. For all $k$, the approximation error is bounded as shown by

$$\text{error bound} = \begin{cases} -\frac{k}{2} - 1 \le \Delta_n \le \frac{k}{2} - 1 & \text{even } k \\ \lfloor -\frac{k}{2} \rfloor \le \Delta_n \le \lfloor \frac{k}{2} \rfloor & \text{odd } k \end{cases} \tag{7}$$
where $\Delta_n = pkt_{new}(k) - \sum_{n=1}^{k} w(n)$.

For a given default MSS of 536Bytes and a *ssthresh* of 65kBytes [7], it is enough to limit attention to $k$ in the range 1 to 14, because this is the range of interest for the slow start phase of TCP. Furthermore, if the TCP scale option is used, then the approximation error caused by (2), given by [1], will grow exponentially. Yet the approximation error caused by (6), proposed in this note, is still linearly bounded as expressed by (7). We expect the average error to be well below $k/2$ for rounds $k$ greater than 15.

## III. New Sum Formula Integration into Model

Sikdar et al [1] use their sum formula for derivations of further results. In this section we compare the formulae for the key result of finding the length of a slow start phase. This result is used throughout their findings.

The number of rounds required to transmit $N$ packets, based on their formula (2), is given by [1] as

$$r(N) = \left\lceil 2\log_2\left(\frac{2N + 4 + 3\sqrt{2}}{2\sqrt{2} + 3\,(2)^{\frac{5}{8}}}\right)\right\rceil. \tag{8}$$

This equation is derived from (2). Therefore we require a model to predict the number of rounds required to transmit $N$ packets based on (6), to integrate our proposed new sum into the model. We note that (6) does not lend itself to a simple inversion due to having both exponential and linear components. However, (6) can be rearranged to give a recursive definition for the required variable $k$. Therefore, to find the number of rounds required to transmit $N = pkt_{new}(k)$ packets using our formula (6) we propose the following recursive algorithm (RA):

1. $n \leftarrow 1$,
2. $k_n \leftarrow 0$,
3. $k_{n+1} \leftarrow 2\log_2\left(\frac{N+2+3/\sqrt{2}+7/200+k_n/2}{2+3/\sqrt{2}}\right)$.
4. $n \leftarrow n+1$,
5. If convergence criterion OK then go to Step 6, else go to Step 3.
6. Finish.

This prescribes an algorithmic method to obtain $k$ precisely. Furthermore, this algorithm can be proven to converge.

*Proof:* **Convergence Proof**

Given $a = (2+3/\sqrt{2}+7/200)/(2+3/\sqrt{2})$ and $b = 1/(2+3/\sqrt{2})$, the iterations of the algorithm described produce the following monotonically increasing sequence

$$\begin{aligned} &2\log_2(a+bN), 2\log_2(a+bN+b\log_2(a+bN)), \\ &2\log_2(a+bN+b\log_2(a+bN+b\log_2(a+bN)),\ldots \end{aligned} \tag{9}$$

To prove that it converges to a limit, it suffices to show that it is bounded above. We note that $e^x > 1 + x$ for $x > 0$ and so we get $x > \ln(1+x)$. By replacing $x$ with $y-1$, we get

$$\ln y < y - 1 \qquad \text{for } y > 1. \tag{10}$$

By RA procedure we have

$$k_n = 2\log_2\left(a + bN + \frac{b}{2}k_{n-1}\right). \tag{11}$$

Using (10), (11) leads to

$$\begin{aligned} k_n &\leq \frac{2}{\ln 2}\left(a + bN + \frac{b}{2}k_{n-1} - 1\right) \\ &= \frac{2}{\ln 2}(a + bN - 1) + \frac{b}{\ln 2}k_{n-1}. \end{aligned} \tag{12}$$

Consequently we get

$$\begin{aligned} k_n &\leq \frac{2}{\ln 2}(a + bN - 1) \\ &\quad + \frac{b}{\ln 2}\left(\frac{2}{\ln 2}(a + bN - 1) + \frac{b}{\ln 2}k_{n-2}\right) \\ &= \frac{2}{\ln 2}(a + bN - 1) + \frac{2b}{(\ln 2)^2}(a + bN - 1) \\ &\quad + \left(\frac{b}{\ln 2}\right)^2 k_{n-2} \\ &= (a + bN - 1)\left[\frac{2}{\ln 2} + \frac{2b}{(\ln 2)^2} + \frac{2b^2}{(\ln 2)^3}\right. \\ &\quad \left. + \cdots + \frac{2b^{n-2}}{(\ln 2)^{n-1}}\right] + \left(\frac{b}{\ln 2}\right)^{n-1} k_1. \end{aligned} \tag{13}$$

Therefore, $k_n$ is bounded by a geometric series with a common ratio of $b/\ln 2 < 1$. In the limit $n \to \infty$, $k_n$ is bounded above by a constant, as shown by

$$\lim_{n\to\infty} k_n \leq \frac{2}{\ln 2 - b}(a + bN - 1). \tag{14}$$

Therefore, the $k_n$ series converges. ∎

Since we are only interested in integer rounds $k$, the ceiling of three iterations of the algorithm is normally enough to provide adequate accuracy. Therefore we obtain

$$\begin{aligned} r_{new}(N) &= \lceil 2\log_2(a + bN + b\log_2(a + bN \\ &\quad + b\log_2(a + bN)))\rceil \\ a &= (2 + 3/\sqrt{2} + 7/200)/(2 + 3/\sqrt{2}) \\ b &= 1/(2 + 3/\sqrt{2}). \end{aligned} \tag{15}$$

Compared with the trial and error formulae (2) and (8) given by [1], our results (6) and (15) are analytically obtained from (1), providing a solid theoretic background. These formulae are more faithful to the window increase pattern of (1), see Table II, where boldface indicates the first incorrect result of corresponding formulae.

TABLE II

COMPARISON OF CALCULATED, FORMULA, AND PROPOSED FORMULA FOR NUMBER OF ROUNDS TO TRANSMIT $N$ PACKETS

| $N$ | Calculated $r(N)$ | $r(N)$ | $r_{new}(N)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 |
| 3 | 2 | 2 | 2 |
| 4 | 3 | 3 | 3 |
| 16 | 5 | 5 | 5 |
| 17 | 6 | 6 | 6 |
| 38 | 7 | 7 | 7 |
| 39 | 8 | 8 | 8 |
| 55 | 8 | 8 | 8 |
| 56 | 8 | **9** | 8 |
| 57 | 8 | 9 | 8 |
| 58 | 9 | 9 | 9 |
| 115 | 10 | 10 | 10 |
| 116 | 10 | 11 | 10 |
| 122 | 10 | 11 | 10 |
| 123 | 11 | 11 | 11 |
| 950 | 16 | 16 | 16 |
| 951 | 16 | 17 | 16 |
| 1042 | 16 | 17 | 16 |
| 1043 | 16 | 17 | **17** |
| 1044 | 17 | 17 | 17 |

## IV. EFFECT OF NEW SUM FORMULAE

The models presented thus far are used to determine the latency of transmitting $N$ packets in slow start. However, they also determine the conditions for recovery after a loss indication.

A correlated loss model, used to model a bursty loss behaviour, was first introduced in [3], and is adopted by [1]. The model states that a packet is lost independently of packets in other rounds, but losses are correlated within one round, such that if a packet is lost, all remaining packets of that round are also lost. Reference [1] uses this model to determine how many packets are successfully received in the following round, which then form the duplicate acknowledgements that become a loss indication through Fast Retransmit [4].

Given that the $i^{\text{th}}$ packet is lost, for the case where the congestion window, *cwnd*, is not limited by the receiver advertised window, $W_{max}$, [1] present the following formulae

$$cwnd_i^0 = w(r(i)) \tag{16}$$

$$n_{max}(i) = pkt(r(i)) \tag{17}$$

$$nloss = n_{max}(i) - i + 1 \tag{18}$$

$$cwnd_i^1 = cwnd_i^0 + \left\lceil \frac{cwnd_i^0 - nloss}{2} \right\rceil \tag{19}$$

$$nrnd_i^1 = cwnd_i^1 - nloss, \tag{20}$$

where $cwnd_i^0$ is the congestion window of round $r(i)$, $n_{max}(i)$ is the sequence number of the last packet sent in round $r(i)$, $nloss$ is the number of losses, and $cwnd_i^1$ and $nrnd_i^1$ are, respectively, the congestion window and number of packets sent in the following round.

As can be seen in Table II, $i = 56$ is the first erroneous value for $r(i)$. Since $r(55)$ gives round $k = 8$ and $r(56)$ gives $k = 9$, this indicates that $i = 56$ is the first packet to be transmitted in round 9. Therefore, if this packet is lost, then the correlated loss model dictates that all packets following in that round are also lost. This should indicate that $nloss = cwnd_i^0$ and consequently $nrnd_i^0 = 0$, indicating that there should be no duplicate acknowledgements to trigger a Fast Retransmit. However, as an erroneous $r(56) = 9$ is given, $w(9)$ gives a value too large, which incorrectly results in $nrnd_i^1 = 3$, such that the minimum number of duplicate acknowledgements will arrive to trigger Fast Retransmit [4]. This behaviour is also present for $i = 57, 81, 82, 116, 117, 165, 166, \ldots$. However, if using our proposed models of (6) and (15), this behaviour is only seen for $i > 1042$, a much larger correct range. Also, $cwnd > 308$ when $i > 1042$, such that the probability for $nloss > cwnd - 2$ is very low, less that 0.65%.

Importantly, this behaviour occurs precisely when the number of losses, $nloss$, should be great enough to cause a time out and not a Fast Retransmit. Therefore, for $i > 55$ the use of (2) and (8) eliminate the possibility for a timeout to occur from a loss indication in slow start. Furthermore, since time outs are a minimum of one second, and are always larger than the round trip time [4], they have significant impact on total flow latency, and hence must be included for an accurate model.

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