

# Minimum Cost Homomorphisms to Oriented Cycles with Some Loops

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## Abstract

For digraphs  $D$  and  $H$ , a homomorphism of  $D$  to  $H$  is a mapping  $f: V(D) \rightarrow V(H)$  such that  $uv \in A(D)$  implies  $f(u)f(v) \in A(H)$ . Suppose  $D$  and  $H$  are two digraphs, and  $c_i(u)$ ,  $u \in V(D)$ ,  $i \in V(H)$ , are nonnegative real costs. The cost of the homomorphism  $f$  of  $D$  to  $H$  is  $\sum_{u \in V(D)} c_{f(u)}(u)$ . The minimum cost homomorphism for a fixed digraph  $H$ , denoted by  $\text{MinHOM}(H)$ , asks whether or not an input digraph  $D$ , with nonnegative real costs  $c_i(u)$ ,  $u \in V(D)$ ,  $i \in V(H)$ , admits a homomorphism  $f$  to  $H$  and if it admits one, find a homomorphism of minimum cost.

The minimum cost homomorphism problem seems to offer a natural and practical way to model many optimization problems such as list homomorphism problems, retraction and precolouring extension problems, chromatic partition optimization, and applied problems in repair analysis.

Our interest is in proving a dichotomy for minimum cost homomorphism problem: we would like to prove that for each digraph  $H$ ,  $\text{MinHOM}(H)$  is polynomial-time solvable, or NP-hard. We say that  $H$  is a digraph with some loops, if  $H$  has at least one loop. For reflexive digraphs  $H$  (every vertex has a loop) the complexity of  $\text{MinHOM}(H)$  is well understood. In this paper, we obtain a full dichotomy for  $\text{MinHOM}(H)$  when  $H$  is an oriented cycle with some loops. Furthermore, we show that this dichotomy is a remarkable progress toward a dichotomy for oriented graphs with some loops.

*Keywords:* homomorphism, minimum cost homomorphism, oriented cycles, NP-hardness, dichotomy.

## 1 Introduction

For digraphs  $D$  and  $H$ , a homomorphism of  $D$  to  $H$  is a mapping  $f: V(D) \rightarrow V(H)$  such that  $uv \in A(D)$  implies  $f(u)f(v) \in A(H)$ . Let  $H$  be a fixed digraph. The *homomorphism problem* for  $H$ , denoted  $\text{HOM}(H)$ , asks whether a digraph  $D$  admits a homomorphism to  $H$ . The *list homomorphism problem* for  $H$ , denoted  $\text{ListHOM}(H)$ , asks whether an input digraph  $D$  with lists (sets)  $L_u \subseteq V(H)$ ,  $u \in V(D)$  admits a homomorphism  $f$  to  $H$  in which  $f(u) \in L_u$  for each  $u \in V(D)$ .

Suppose  $D$  and  $H$  are two digraphs, and  $c_i(u)$ ,  $u \in V(D)$ ,  $i \in V(H)$ , are nonnegative real costs. The cost of the homomorphism  $f$  of  $D$  to

$H$  is  $\sum_{u \in V(D)} c_{f(u)}(u)$ . The minimum cost homomorphism for a fixed digraph  $H$ , denoted by  $\text{MinHOM}(H)$ , asks whether or not an input digraph  $D$ , with nonnegative real costs  $c_i(u)$ ,  $u \in V(D)$ ,  $i \in V(H)$ , admits a homomorphism  $f$  to  $H$  and if it admits one, find a homomorphism of minimum cost.

The minimum cost homomorphism problem seems to offer a natural and practical way to model many optimization problems such as list homomorphism problems (Hell 2004), retraction and precolouring extension problems (Hell 2004, Marx 2004, 2006), chromatic partition optimization (Halldorsson 2001, Jansen 2000, Jiang 1999) (which itself has a number of well-studied special cases and applications (Kroon 1996, Supowit 1987)), and applied problems in repair analysis (Gutin 2006).

Let  $D$  be a digraph. We say that  $u$  dominates  $v$  if  $uv \in A(D)$ . If  $uv$  is an arc of  $D$ , we say that  $u$  is an *in-neighbor* of  $v$  and  $v$  is an *out-neighbor* of  $u$ . The number of in-neighbors (out-neighbors) of  $v$  is called the *in-degree* (*out-degree*) of  $v$ . A vertex  $u \in V(D)$  is a *source* (*sink*), if  $u$  has in-degree (out-degree) zero. A vertex  $u \in V(D)$  has a loop, if  $uu \in A(D)$ . We call  $D$  *irreflexive*, if no vertex of  $D$  has a loop. If  $D$  has a loop at every vertex, then  $D$  is reflexive. We say that  $D$  is a *digraph with some loops*, if at least one vertex of  $D$  has a loop. When we wish to stress that a family of digraphs may contain digraphs with some loops, we will speak of *digraphs with possible loops*. Finally, we denote by  $I(D)$  the irreflexive digraph  $D'$  obtained from a digraph  $D$  with possible loops by removing all existing loops.

Our interest is in proving a *dichotomy* for minimum cost homomorphism problem: we would like to prove that for each digraph  $H$  with possible loops,  $\text{MinHOM}(H)$  is polynomial-time solvable, or NP-hard. Very recently, Gutin, Rafiey and Yeo (2008) conjectured that such a dichotomy exists. (cf. Section 2.)

There has been a lot of interest (Feder 1999, Bulatov 2003, 2005, Feder 1999, Hell 1990, Larose 2004, Maróti 2006) for more than a decade to prove dichotomies for different derivatives of homomorphism problem such as  $\text{HOM}(H)$ , and  $\text{ListHOM}(H)$ . In particular for  $\text{ListHOM}(H)$ , Bulatov has shown that a dichotomy exists (Bulatov 2003); for  $\text{HOM}(H)$ , it is still an open problem whether there exists a dichotomy. However, there are several conjectures (Feder 1999, Larose 2004, Bulatov 2005) in the literature addressing such a dichotomy. Moreover, Feder and Vardi have shown that proving a dichotomy for  $\text{HOM}(H)$  when  $H$  is a general digraph, would imply that the well-known Constraint Satisfaction Problem Dichotomy Conjecture also holds (Feder 1999).

It is easy to see that each  $\text{HOM}(H)$  is a special case of  $\text{ListHOM}(H)$ , i.e., is polynomial time reducible to  $\text{ListHOM}(H)$ , obtained by setting all lists to  $V(H)$ . Similarly,  $\text{MinHOM}(H)$  generalizes  $\text{ListHOM}(H)$  by

setting  $c_i(u) = 0$  if  $i \in L_u$  and  $c_i(u) = 1$  otherwise. Let us assume a hierarchy which has  $\text{HOM}(H)$  at the lowest level,  $\text{ListHOM}(H)$  at the middle, and  $\text{MinHOM}(H)$  at the highest level. It is worth noting that knowing a dichotomy for any of this problem does not immediately imply a dichotomy for the other ones. However, we can always use the tractable cases of each problem in the higher levels of hierarchy to find tractable cases in the lower levels and we are also able to use the NP-hard cases in the lower levels to explore NP-hard cases in the higher levels.

For undirected graphs  $H$ , a dichotomy classification for the problem  $\text{MinHOM}(H)$  has been provided in (Gutin, Hell, Rafiey and Yeo 2008). Thus, the minimum cost homomorphism problem for graphs has been handled, and interest shifted to directed graphs. The first studies (Gutin 2006, 2007) focused on irreflexive digraphs, where dichotomies have been obtained for digraphs  $H$  such that  $H$  is a semicomplete or semicomplete multipartite digraph. Some other dichotomies have been also provided for quasi-transitive digraphs (Gupta 2007) and locally in-semicomplete digraphs (Gupta, Karimi, Kim and Rafiey 2008) as two important generalizations of semicomplete digraphs. The authors of (Gutin & Kim 2007) generalized the result of (Gutin 2006) for semicomplete digraphs to semicomplete digraphs with possible loops, and also raised a conjecture for reflexive digraphs. Gupta, Hell, Karimi and Rafiey (2008) verified this conjecture for reflexive digraphs and obtained a full dichotomy for  $\text{MinHOM}(H)$  when  $H$  is a reflexive digraph.

Homomorphisms to oriented cycles have been investigated in a number of papers (Gutjahr 1991, Hell 1995, Feder 2001, Gutin, Rafiey and Yeo 2008). In particular, Feder has provided a dichotomy for  $\text{HOM}(H)$  when  $H$  is an oriented cycle (Feder 2001). The other important result is a  $\text{MinHOM}$  dichotomy obtained for oriented cycles (Gutin, Rafiey and Yeo 2008). We will return to this dichotomy in Section 2.

In this paper, we obtain a full dichotomy for  $\text{MinHOM}(H)$  when  $H$  is an oriented cycle with some loops. Furthermore, we show that this dichotomy is a remarkable progress toward a dichotomy for oriented graphs with some loops. The paper is organized as follows. The next section has been allocated to terminology and preliminaries. In Section 3, we provide a dichotomy for oriented cycles with some loops. In the last section, we discuss why this dichotomy is important and how it might play an important role in finding a dichotomy for oriented graphs with some loops. We will finish this paper with introducing an open problem.

## 2 Terminology and Preliminaries

We begin this section with introducing two digraph properties which play the main role in all  $\text{MinHOM}$  dichotomies proved for different classes of digraphs. These two properties are  $\text{Min-Max}$  ordering and  $k$ - $\text{Min-Max}$  ordering and are defined as follows.

Let  $H$  be a digraph with possible loops and  $<$  a linear ordering of  $V(H)$ . Two arcs  $ab, cd \in A(H)$  are called a *crossing pair* if  $a < c$ , and  $d < b$ .

**Definition 2.1** *A linear ordering  $<$  of  $V(H)$  is a  $\text{Min-Max}$  ordering if, for each crossing pair  $ab, cd \in A(H)$  we have  $ad, cb \in A(H)$ .*

Clearly, if  $H$  has no crossing pair then  $<$  is a  $\text{Min-Max}$  ordering.

**Definition 2.2** *Let  $k \geq 2$  be an integer. A digraph  $H$  admits a  $k$ - $\text{Min-Max}$  ordering if the following conditions hold:*

- $H$  admits a homomorphism  $f$  to a directed  $k$ -cycle  $0, 1, \dots, k-1, 0$ , i.e., every arc of  $H$  is an arc from  $V_i = f^{-1}(i)$  to  $V_{i+1} = f^{-1}(i+1)$  for some  $i \in \{0, 1, \dots, k-1\}$ , and
- there is a linear ordering  $<$  of vertices of each  $V_i = f^{-1}(i)$ , so that for each crossing pair  $ab, cd \in A(H)$  ( $a < c$ , and  $d < b$ ) where  $a, c \in V_i$ , and  $b, d \in V_{i+1}$  we have  $ad, cb \in A(H)$ ,

where all indices  $i+1$  are taken modulo  $k$ .

It turns out that if a digraph  $H$  has a  $\text{Min-Max}$  ordering or  $k$ - $\text{Min-Max}$  ordering for some  $k \geq 2$ ,  $\text{MinHOM}(H)$  is polynomial time solvable (Gutin 2006, 2007). Very recently, (Gutin, Rafiey and Yeo 2008) conjectured that all digraphs  $H$  for which  $\text{MinHOM}(H)$  is polynomial time solvable, should have a  $\text{Min-Max}$  ordering or a  $k$ - $\text{Min-Max}$ .

**Conjecture 2.3** (Gutin, Rafiey and Yeo 2008) *Let  $H$  be a digraph with possible loops. Then  $\text{MinHOM}(H)$  is polynomial time solvable if  $H$  has a  $\text{Min-Max}$  ordering or a  $k$ - $\text{Min-Max}$  ordering for some  $k \geq 2$ . Otherwise,  $\text{MinHOM}(H)$  is NP-hard.*

Clearly, It is the NP-hardness part of this conjecture which is the open part of it. We remark that the NP-hardness part of this conjecture can be shown, if one gives a nice characterization of digraphs with  $\text{Min-Max}$  and  $k$ - $\text{Min-Max}$  orderings. We note that, in particular, if these digraphs can be characterized by a few forbidden induced subgraphs, then the NP-hardness part easily follows. Indeed, it is sufficient to prove that minimum cost homomorphism is NP-hard for all these induced subgraphs due to the following fact (Gutin 2006).

**Proposition 2.4** (Gutin 2006) *Let  $H$  be a digraph with possible loops and  $H'$  be an induced subgraph of  $H$ . If  $\text{MinHOM}(H')$  is NP-hard, then  $\text{MinHOM}(H)$  is NP-hard.*

It follows from the definition of  $k$ - $\text{Min-Max}$  ordering that a digraph  $H$  with some loops, can not admit a  $k$ - $\text{Min-Max}$  ordering for some  $k \geq 2$ . This leads us to a simpler form of the  $\text{MinHOM}$  conjecture for digraphs with some loops.

**Conjecture 2.5** *Let  $H$  be a digraph with some loops. Then  $\text{MinHOM}(H)$  is polynomial time solvable if  $H$  admits a  $\text{Min-Max}$  ordering. Otherwise,  $\text{MinHOM}(H)$  is NP-hard.*

In this paper, we verify Conjecture 2.5 for oriented cycles with some loops. Let us now define oriented cycles. We follow (Hell 1995) for the definitions of oriented cycles and paths. Let  $[p]$  denote the set  $\{0, 1, \dots, p\}$ . An *oriented path*  $P$  is a sequence of distinct vertices  $[b_0, b_1, \dots, b_p]$ , such that, for each  $i \in \{0, 1, \dots, p-1\}$ , either  $b_i b_{i+1} \in A(P)$  (a *forward* arc of  $P$ ) or  $b_{i+1} b_i \in A(P)$  (a *backward* arc of  $P$ ), and  $P$  has no other arcs. The direction in which  $P$  is traversed is emphasized by saying that  $b_0$  is the *initial vertex* of  $P$ , and  $b_p$  is the *terminal vertex* of  $P$ , respectively.

Let  $P = b_0 b_1 \dots b_p$  be an oriented path. We assign *levels* to the vertices of  $P$  as follows: we set  $l(b_0) = 0$ , and  $l(b_{t+1}) = l(b_t) + 1$ , if  $b_t b_{t+1}$  is forward and  $l(b_{t+1}) = l(b_t) - 1$ , if  $b_t b_{t+1}$  is backward. We say that  $P$  is of *type  $r$*  if  $r = \max\{l(b_i) : i \in [p]\} = l(b_p)$  and  $0 \leq l(b_t) \leq r$  for each  $t \in [p]$ .

The following proposition was first proved in (Häggkvist 1988); see also (Feder 2001, Zhu 1992) and Lemma 2.36 in (Hell 2004).

**Proposition 2.6** Let  $P_1$  and  $P_2$  be two oriented paths of type  $r$ . Then there is an oriented path  $P$  of type  $r$  that maps homomorphically to  $P_1$  and  $P_2$  such that the initial vertex of  $P$  maps to the initial vertices of  $P_1$  and  $P_2$  and the terminal vertex of  $P$  maps to the terminal vertices of  $P_1$  and  $P_2$ . The length of  $P$  is polynomial in the lengths of  $P_1$  and  $P_2$ .

We will use the following notation in this paper:  $L(P) = \min\{l(b_j) : j \in [p]\}$ ,  $H(P) = \max\{l(b_j) : j \in [p]\}$ ,  $V_L(P) = \{b_t : l(b_t) = L(P), t \in [p]\}$ , and  $V_H(P) = \{b_t : l(b_t) = H(P), t \in [p]\}$ .

An *oriented cycle*  $C$  is a digraph obtained from an oriented path  $P$  by identifying its initial and terminal vertices. Thus an oriented cycle  $C$  can be given by a circular sequence of vertices  $[b_0, b_1, \dots, b_p, b_0]$ , such that, for each  $i \in \{0, 1, \dots, p\}$ , either  $b_i b_{i+1} \in A(C)$  (a *forward arc* of  $C$ ) or  $b_{i+1} b_i \in A(C)$  (a *backward arc* of  $C$ ), and  $C$  has no other arcs. (Subscript addition is taken modulo  $p$ .) Since we do not distinguish an initial vertex of an oriented cycle, we usually choose the most convenient vertex to start listing  $C$ . In what follows, we will always consider the direction  $b_0 b_1 \dots b_p b_0$  in which the number of forward arcs is not smaller than the number of backward arcs. This way, the *net length* of  $C$  is the difference between the number of forward arcs and the number of backward arcs and hence is always nonnegative. Let  $C$  be an oriented cycle with possible loops. The net length of  $C$ , denoted  $\lambda(C)$ , is equal to the net length of  $I(C)$ .

Let  $C$  be an oriented cycle. We can assign *levels* to the vertices of  $C$  as follows:  $l(b_0) = k$ , where  $k$  is an integer, and  $l(b_{t+1}) = l(b_t) + 1$ , if  $b_t b_{t+1}$  is forward and  $l(b_{t+1}) = l(b_t) - 1$ , if  $b_t b_{t+1}$  is backward. Clearly, the value of each  $l(b_i)$ ,  $i \in [p]$ , depends on both  $k$  and the choice of the initial vertex  $b_0$ . We refer to  $P_{b_0}^C$  as the oriented path  $b_0 b_1 \dots b_p b_0$  obtained from the oriented cycle  $C = b_0 b_1 \dots b_p b_0$  such that the first  $b_0$  and the last  $b_0$  are distinct vertices. (We open  $C$  on  $b_0$ .) This way, each vertex of  $P_{b_0}^C$  has a unique counterpart in  $C$ . So, when we refer to a vertex in  $P_{b_0}^C$ , one can imagine its corresponding vertex in  $C$ .

The following notation is extensively used in the rest of this paper:  $L(C) = \min\{l(b_j) : j \in [p]\}$ ,  $H(C) = \max\{l(b_j) : j \in [p]\}$ ,  $V_L(C) = \{b_t : l(b_t) = L(C), t \in [p]\}$ , and  $V_H(C) = \{b_t : l(b_t) = H(C), t \in [p]\}$ .

A *directed cycle* (respectively, a *directed path*) is an oriented cycle (respectively, oriented path) in which all edges are in the same direction. We denote a directed cycle (respectively, path) with  $k$  vertices by  $\vec{C}_k$  (respectively,  $\vec{P}_k$ ). An *oriented graph* is a digraph with no cycle of length two  $\vec{C}_2$ .

An oriented cycle  $C$  is *balanced* if its net length is zero, otherwise  $C$  is *unbalanced*. Note that if  $C$  is balanced, the vertices of  $C$  that belong to  $V_L(C)$  and  $V_H(C)$ , do not change by changing  $b_0$  and  $k$ . We say that a balanced oriented cycle  $C$  is *of the form*  $(l^+ h^+)^q$  with  $q \geq 1$ , if there is an initial vertex  $b_0 \in V_L(C)$  such that  $P = C - b_p b_0$  (for  $C = b_0 b_1 \dots b_p b_0$ ) can be written as  $P = x_1 P_1 y_1 R_1 x_2 P_2 y_2 R_2 \dots x_q P_q y_q R_q$ , where  $x_i \in V_L(C)$ ,  $y_i \in V_H(C)$  for each  $i \in [q]$ , and  $P_i, R_i$  are oriented paths such that vertices in  $V_L(C) \cap V(P_i)$  ( $V_H(C) \cap V(R_i)$ ) appear before all vertices in  $V_H(C) \cap V(P_i)$  ( $V_L(C) \cap V(R_i)$  in  $R_i$ ) for each  $i \in [q]$  (Gutin, Rafiey and Yeo 2008). We write  $l^+ h^+$  instead of  $(l^+ h^+)^1$ . (see Figure 1 for an example of balanced oriented cycles of the form  $l^+ h^+$ .)

In a balanced oriented cycle  $C$  of the form  $l^+ h^+$ , a vertex  $b_0 \in V_L(C)$  is the *absolute base*, if there is no vertex  $u \in V_L(C)$  between  $b_0$  and the first vertex of

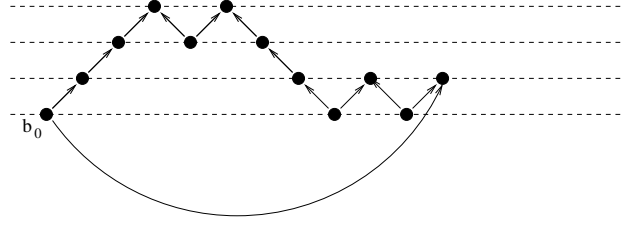


Figure 1: A balanced oriented cycle of the form  $l^+ h^+$ .  $b_0$  is the absolute base.

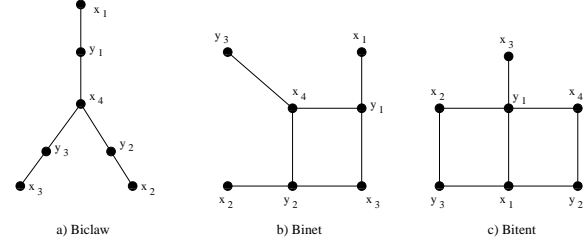


Figure 2: The biclaw, the binet, and the bitent.

$V_H(C)$  in the direction  $b_0 b_1 \dots b_p b_0$ . Correspondingly, the path  $P_{b_0}^C = b_0 b_1 \dots b_p b_0$  is called the *absolute base path*.

Consider the oriented cycle with the vertex set  $C_4^0 = \{1, 2, 3, 4\}$ , and the arc set  $\{12, 32, 14, 34\}$ . The next theorem follows from the main result of (Gutin, Rafiey and Yeo 2008).

**Theorem 2.7** (Gutin, Rafiey and Yeo 2008) Let  $C$  be an irreflexive oriented cycle.

- If  $C$  has net length  $k \geq 2$ , then it has a  $k$ -Min-Max ordering and  $\text{MinHOM}(C)$  is polynomial time solvable.
- If  $C$  has net length  $k = 1$ , then it has a Min-Max ordering and  $\text{MinHOM}(C)$  is polynomial time solvable.
- If  $C$  is balanced of the form  $l^+ h^+$  or  $C = C_4^0$ , then  $C$  has a Min-Max ordering and  $\text{MinHOM}(C)$  is polynomial-time solvable. For all other balanced oriented cycles  $C$ ,  $\text{MinHOM}(C)$  is NP-hard.

Finally, we denote by  $B(H)$  the bipartite graph obtained from  $H$  as follows. Each vertex  $v$  of  $H$  gives rise to two vertices of  $B(H)$  - a *white* vertex  $v'$  and a *black* vertex  $v''$ ; each arc  $vw$  of  $H$  gives rise to an edge  $v'w''$  of  $B(H)$ . Let *biclaw*, *binet*, and *bitent* be the digraphs, shown in Figure 2. The following proposition in (Gutin, Hell, Rafiey and Yeo 2008) is a useful tool to prove some NP-hard cases in this paper.

**Proposition 2.8** Let  $H$  be a digraph. If  $B(H)$  contains a biclaw, or a binet, or a bitent as an induced subgraph, then  $\text{MinHOM}(H)$  is NP-hard.

### 3 Dichotomy

In this section, we provide a full dichotomy for  $\text{MinHOM}(C)$  when  $C$  is an oriented cycle with some loops. To do that, first of all, we partition the class of oriented cycles with some loops to three subclasses: the first contains all oriented cycles with some loops  $C$  such that the net length  $\lambda(C)$  is more than one; the second contains all  $C$  such that  $\lambda(C) = 1$ ; the third

contains all  $C$  such that  $I(C)$  is balanced. We will verify the Conjecture 2.5 for each of these subclasses separately.

### 3.1 Tools

The next Lemma has been proved in (Gupta, Karimi, Kim and Rafiey 2008).

**Lemma 3.1** *Let  $D$  and  $H$  be two digraphs. Suppose  $D$  and  $H$  have two pairs of vertices  $u, v$  and  $x, z$ , respectively such that:*

- (a) *there is a homomorphism  $f_1$  of  $D$  to  $H$  which maps both  $u$  and  $v$  to  $z$ ;*
- (b) *there is no homomorphism of  $D$  to  $H$  which maps both  $u$  and  $v$  to  $x$ ;*
- (c) *there is a homomorphism  $f_2$  of  $D$  to  $H$  which maps  $u$  to  $x$  and  $v$  to  $z$ ;*
- (d) *there is a homomorphism  $f_3$  of  $D$  to  $H$  which maps  $v$  to  $x$  and  $u$  to  $z$ ;*

*Then  $\text{MinHOM}(H)$  is NP-hard.*

The following lemmas are useful tools in proving the NP-hard cases.

**Lemma 3.2** *Let  $C$  be an oriented cycle with some loops having a loop on an arbitrary vertex  $z$  and let  $D$  be a digraph. Suppose  $D$  has a pair of distinct vertices  $u, v$ , and  $C$  has a pair of not necessarily distinct vertices  $x, y$ , distinct from  $z$  such that:*

- (a) *there is a homomorphism  $f_1$  of  $D$  to  $C$  which maps both  $u$  and  $v$  to  $z$ ;*
- (b) *there is no homomorphism of  $D$  to  $C$  which maps  $u$  to  $x$  and  $v$  to  $y$ ;*
- (c) *there is a homomorphism  $f_2$  of  $D$  to  $C$  which maps  $u$  to  $x$  and  $v$  to  $z$ ;*
- (d) *there is a homomorphism  $f_3$  of  $D$  to  $C$  which maps  $u$  to  $z$  and  $v$  to  $y$ ,*

*Then  $\text{MinHOM}(C)$  is NP-hard.*

**Proof:** If  $x$  and  $y$  are not distinct then by Lemma 3.1,  $\text{MinHOM}(C)$  is NP-hard. Thus, we assume that  $x$  and  $y$  are distinct vertices. This way, there should be a vertex  $w$  in  $C$  such that there are two internally disjoint oriented paths  $P = ww_1 \dots w_n z$ , and  $Q = ww'_1 \dots w'_m z$  from  $w$  to  $z$  in  $C$ , and at least one of  $x$  and  $y$  is only in one of these oriented paths. Without loss of generality assume that this vertex is  $x$ , which is in  $P$ , and we have  $x = w_i, 1 \leq i \leq n$ . Note that  $w$  may be equal to  $y$  if  $x$  and  $y$  are adjacent. If  $w$  and  $y$  are distinct vertices then we may assume that  $y = w'_j, 1 \leq j \leq m$  in  $Q$ .

Now, We will construct a polynomial time reduction from the maximum independent set problem to  $\text{MinHOM}(C)$ . Let  $G$  be an arbitrary undirected graph. We replace every edge  $u'v' \in E(G)$  by the digraph  $D'$  consisting of a set of special vertices  $\{u', v', r, s, t\}$ , and a set of digraphs between some pairs of these vertices as follows:

- There is an oriented path  $u'u_1 \dots u_{i-1}s$  from  $u'$  to  $s$ , which is exactly isomorphic to the oriented path from  $w$  to  $x$  in  $P$ .
- There is an oriented path  $su_{i+1} \dots u_n t$  from  $s$  to  $t$ , which is exactly isomorphic to the oriented path from  $x$  to  $z$  in  $P$ .

- If  $w$  and  $y$  are distinct vertices then there is an oriented path  $v'v_1 \dots v_{j-1}r$  from  $v'$  to  $r$ , which is exactly isomorphic to the oriented path from  $w$  to  $y$  in  $Q$ . If  $w = y$  then  $v' = r$  and there is no oriented path between them.

- If  $w$  and  $y$  are distinct vertices then there is an oriented path  $rv_{j+1} \dots v_m t$  from  $r$  to  $t$ , which is exactly isomorphic to the oriented path from  $y$  to  $z$  in  $Q$ . If  $w = y$  then  $v' = r$ , and there is an oriented path  $rv_1 \dots v_m t$  from  $r$  to  $t$ , which is isomorphic to  $Q$ .

- There is a digraph  $D_1$  between  $s$  and  $r$ , which is isomorphic to  $D$ , and this isomorphism maps  $s$  to  $u$  and  $r$  to  $v$ .

For simplicity, let us rename  $s$  to  $u_i$  and  $r$  to  $v_j$ . We will denote this digraph obtained from  $G$  by  $D''$ .

We assign the costs as follows:

$c_z(a) = 1, c_w(a) = 0$  for  $a \in V(G)$ , and  $c_b(a) = +\infty$  for  $b \in V(C) - \{z, w\}, a \in V(G)$ ;  
 $c_{w_{i'}}(u_{i'}) = c_z(u_{i'}) = 0$  apart from  $c_b(u_{i'}) = +\infty$  for  $b \in V(C) - \{z, w_{i'}\}$ ;  
 $c_{w'_{j'}}(v_{j'}) = c_z(v_{j'}) = 0$  apart from  $c_b(v_{j'}) = +\infty$  for  $b \in V(C) - \{z, w'_{j'}\}$ ;

$c_b(a) = 0$  for  $b \in V(C), a \in V(D_1) - \{u_i, v_j\}$ ;

There is always a homomorphism of finite cost from  $D''$  to  $C$ . (We can map all vertices of  $D''$  to  $z$ .) Let  $f$  be a homomorphism of  $D''$  to  $C$  with finite cost and let  $S = \{u' \in V(G) : f(u') = w\}$ . Then,  $S$  is an independent set in  $G$  since we cannot assign color  $w$  to both  $u'$  and  $v'$  in  $V(G)$  whenever there is an edge between them. In fact, if  $f(u') = f(v') = w$  then  $f(u_i) = w_i = x$  and  $f(v_j) = w'_j = y$  as the homomorphism has finite cost. Hence,  $f$  is a homomorphism of  $D_1$  (which is isomorphic to  $D$ ) to  $C$  such that it maps  $u_i$  (correspondingly  $u$  in  $D$ ) to  $x$  and  $v_j$  (correspondingly  $v$  in  $D$ ) to  $y$ , contrary to part (b). Observe that a minimum cost homomorphism will assign as many vertices of  $V(G)$  as possible with  $w$ .

Conversely, suppose we have an independent set  $I$  of  $G$ . Then we can build a homomorphism of finite cost  $f$  of  $D''$  to  $C$  such that  $f(u') = w$  for all  $u' \in I$  and  $f(u') = z$  for all  $u' \in V(G) \setminus I$ . To do so, it is enough to show that if there is an edge between  $u'v'$  in  $G$ , there are homomorphisms  $g_1, g_2$ , and  $g_3$  of the gadget  $D'$  to  $C$  such that:

- $g_1(u') = z$  and  $g_1(v') = z$ ;
- $g_2(u') = w$  and  $g_2(v') = z$ ;
- $g_3(u') = z$  and  $g_3(v') = w$ ;

We will build these homomorphisms as follows:

- $g_1(u') = z, g_1(a) = z$  for  $a \in V(D')$ ,  $g_1(v') = z$ ;
- $g_2(u') = w, g_2(u_{i'}) = w_{i'}, g_2(t) = z, g_2(v_{j'}) = z, g_2(v') = z, g_2(a) = f_2(a)$  for  $a \in V(D_1)$ ;
- $g_3(v') = w, g_3(v_{j'}) = w'_{j'}, g_3(t) = z, g_3(u_{i'}) = z, g_3(u') = z, g_3(a) = f_3(a)$  for  $a \in V(D_1)$ ;

Hence, a minimum cost homomorphism  $f$  of  $D'$  to  $C$  yields a maximum independent set of  $G$  and vice versa, which completes the proof.  $\diamond$

**Lemma 3.3** *Let  $C$  be an oriented cycle with some loops having a loop on an arbitrary vertex  $x$  and let  $D$  be a digraph with cost  $c'_i(u), i \in V(C), u \in V(D)$ , where  $c'_i(u)$  is either  $+\infty$  or zero, and there is at least one  $c'_i(u)$  which is  $+\infty$ . If  $D$  has a pair of distinct vertices  $u, v$ , and  $C$  has a pair of not necessarily distinct vertices  $x, y$ , distinct from  $z$ , and the following conditions hold:*

- (a)  $c'_x(u) = 0, c'_y(v) = 0$ ;
- (b) there is a homomorphism  $f_1$  with zero cost of  $D$  to  $C$  which maps both  $u$  and  $v$  to  $z$ ;
- (c) there is no homomorphism with finite cost of  $D$  to  $C$  which maps  $u$  to  $x$  and  $v$  to  $y$ ;
- (d) there is a homomorphism  $f_2$  with zero cost of  $D$  to  $C$  which maps  $u$  to  $x$  and  $v$  to  $z$ ;
- (e) there is a homomorphism  $f_3$  with zero cost of  $D$  to  $C$  which maps  $u$  to  $z$  and  $v$  to  $y$ ,

then  $\text{MinHOM}(C)$  is NP-hard.

**Proof:** We will construct a polynomial time reduction from the maximum independent set problem to  $\text{MinHOM}(C)$ . Let  $G$  be an arbitrary undirected graph. We replace every edge  $u'v' \in E(G)$  by the digraph  $D'$ , introduced in the proof of Lemma 3.2. We will denote this digraph obtained from  $G$  by  $D''$ . the costs are exactly like the costs in the proof of Lemma 3.2, apart from:

$$c_b(a) = c'_b(a) \text{ for } b \in V(C), a \in V(D_1) - \{u_i, v_j\};$$

Note that  $D_1$  is isomorphic to  $D$ , and there is a one to one correspondence between vertices of  $D$  and  $D_1$ . Let  $f$  be a homomorphism of  $D''$  to  $C$  with finite cost and let  $S = \{u' \in V(G) : f(u') = w\}$ . Since, there is no homomorphism of finite costs of  $D_1$  (isomorphic to  $D$ ) to  $C$  which maps  $u$  to  $x$  and  $v$  to  $y$ , then,  $S$  is an independent set in  $G$ .

Conversely, suppose we have an independent set  $I$  of  $G$ . Then we can build a homomorphism of finite cost  $f$  (similar to the proof of Lemma 3.2) of  $D''$  to  $C$  such that  $f(u') = w$  for all  $u' \in I$  and  $f(u') = z$  for all  $u' \in V(G) \setminus I$ .

Hence, a minimum cost homomorphism  $f$  of  $D'$  to  $C$  yields a maximum independent set of  $G$  and vice versa, which completes the proof.  $\diamond$

Consider the oriented cycle  $C_3^1$  with the vertex set  $\{1, 2, 3\}$ , and the arc set  $\{12, 23, 13\}$ , and the reflexive directed cycle  $C_2^2$  with the vertex set  $\{1, 2\}$  and the arc set  $\{11, 22, 12, 21\}$ .

**Lemma 3.4** *Let  $C'$  be an oriented cycle obtained from another oriented cycle  $C$  with possible loops by adding a loop to a vertex  $z$  of  $C$ , which is neither a source nor a sink. If  $I(C') \neq C_3^1$  and  $C' \neq C_2^2$ , then  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** Since  $z$  is neither source nor sink then there is a vertex  $x$ , dominating  $z$ , and a vertex  $y$ , dominated by  $z$  in  $C'$ . Note that  $x = y$  if  $I(C')$  is a directed cycle of length 2. Consider the digraph  $D$  with the vertex set  $\{u, v\}$ , and the arc set  $\{uv\}$ . Then there is no homomorphism from  $D$  to  $C'$ , which maps  $u$  to  $x$  and  $v$  to  $y$  unless  $I(C') = C_3^1$  or  $C' = C_2^2$ , meeting the condition (b) of Lemma 3.2. The following homomorphisms meet the conditions (a), (c), and (d) of Lemma 3.2, respectively:

- $f_1(u) = z$ , and  $f_1(v) = z$ ;
- $f_2(u) = x$ , and  $f_2(v) = z$ ;
- $f_3(u) = z$ , and  $f_3(v) = y$ ;

Hence,  $\text{MinHOM}(C')$  is NP-hard.  $\diamond$

**Lemma 3.5** *Let  $C' \neq C_2^2$  be an oriented cycle obtained from another oriented cycle with possible loops  $C$  by adding a loop to a vertex  $z$  of  $C$ . If  $\text{MinHOM}(C)$  is NP-hard, then  $\text{MinHOM}(C')$  is also NP-hard.*

**Proof:** Let  $C''$  be the directed cycle with the vertex set  $\{1, 2\}$  and the arc set  $\{11, 12, 21\}$ , where  $\text{MinHOM}(C'')$  is NP-hard by Lemma 3.4.  $C'$  has a symmetric arc ( $u$  dominates  $v$  and  $v$  dominates  $u$ ) if and only if  $C' = C_2^2$  or  $C' = C''$ . Since  $\text{MinHOM}(C'')$  is NP-hard, the current lemma is true for  $C' = C''$ . On the other hand, it is trivial to check that  $C$  has a Min-Max ordering and  $\text{MinHOM}(C)$  is polynomial time solvable, when  $I(C') = C_3^1$ ; hence the current lemma is also true for oriented cycles  $C'$  for which  $I(C') = C_3^1$ .

Now, let us assume that  $C'$  is not  $C_2^2, C''$ , and all oriented cycles  $C'$  for which we have  $I(C') = C_3^1$ . If  $z$  is neither a source nor a sink, then  $\text{MinHOM}(C')$  is NP-hard by Lemma 3.4. Thus, we assume that  $z$  is either a source or a sink. without loss of generality assume that it is a sink. Moreover, As we exclude  $C_2^2$  and  $C''$ ,  $I(C')$  (equivalently,  $I(C)$ ) will not have any symmetric arc.

Now, we will construct a polynomial time reduction from  $\text{MinHOM}(C)$  to  $\text{MinHOM}(C')$ . An instance of  $\text{MinHOM}(C)$  contains an input digraph  $D$  with  $n$  vertices and the costs  $c_i(u)$ ,  $u \in V(D), i \in V(C)$ . Let all costs  $c_i(u)$  be bounded from above by  $m$ . We can partition the vertices of  $D$  to four sets as follows:

- $U_1$ , where each vertex  $u \in U_1$  has a loop;
- $U_2$ , where no vertex  $u \in U_2$  has a loop, and no vertex of  $U_2$  is a source or sink in  $D$ ;
- $U_3$ , where no vertex  $u \in U_3$  has a loop, and every vertex of  $U_2$  is a source in  $D$ ;
- $U_4$ , where no vertex  $u \in U_4$  has a loop, and every vertex of  $U_2$  is a sink in  $D$ .

It is easy to check that there is no homomorphism of  $D$  to  $C$  which maps  $u \in U_i, i = 1, 2, 3$  to  $z$  in  $C$ . To make an instance of  $\text{MinHOM}(C')$ , let us keep  $D$  as the input digraph and change the costs as follows:  $c'_b(a) = c_b(a)$  for  $a \in V(D), b \in V(C') - \{z\}$ , and  $c'_z(u) = nm + 1, u \in U_i, i = 1, 2, 3$  apart from  $c'_z(u) = c_z(u), u \in U_4$ . Observe that if  $\text{MinHOM}(C')$  returns a minimum cost homomorphism  $f$  of  $D$  to  $C'$  with a cost more than  $nm$ , then there is no homomorphism from  $D$  to  $C$ . Moreover, if  $\text{MinHOM}(C')$  returns a minimum cost homomorphism  $f$  of  $D$  to  $C'$  with a cost less than  $nm + 1$ , then  $f$  is a minimum cost homomorphism of  $D$  to  $C$  as well. Finally, if there is no homomorphism from  $D$  to  $C'$ , then there is no homomorphism of  $D$  to  $C$ .  $\diamond$

### 3.2 Oriented Cycles $C$ with $\lambda(C) \geq 2$

**Theorem 3.6** *Let  $C'$  be an oriented cycle with some loops such that  $\lambda(C') \geq 2$ . If  $C' = C_2^2$ , then  $\text{MinHOM}(C')$  is polynomial time solvable. Otherwise,  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** It is trivial to see that  $C_2^2$  has a Min-Max ordering. Thus, we will assume that  $C' \neq C_2^2$ . To prove this theorem, it is sufficient by Lemma 3.5 to show that  $\text{MinHOM}(C)$  is NP-hard, where  $C$  is an oriented cycle obtained from  $C'$  by removing all loops but the loop of  $z$ . Since the net length of  $I(C')$  is more than one,  $I(C')$  is not equal to  $C_3^1$ . Thus, if  $z$  is neither a source nor a sink in  $I(C)$ ,  $\text{MinHOM}(C)$  is NP-hard by Lemma 3.4. In what follows, we prove that when  $z$  is either a source or a sink in  $I(C)$ , then  $\text{MinHOM}(C)$  is NP-hard. Without loss of generality, we assume that  $z$  is a sink in  $I(C)$ .

To show that  $\text{MinHOM}(C)$  is NP-hard, we will construct a digraph  $D$ , which meets the conditions of Lemma 3.2. First of all, consider the oriented path  $P_z^C = za_1a_2 \dots a_nz$ . For simplicity, let us show the last  $z$  by  $z'$ , i.e.,  $P_z^C = za_1a_2 \dots a_nz'$ . It follows from the definition of  $P_z^C$  and the net length of  $I(C)$  that  $l(z') - l(z) \geq 2$ . Hence, we will always have a vertex  $x \neq z$  in  $P_z^C$  such that  $l(x) - l(z) = 1$ . Among such vertices, we will choose  $x$  as the first vertex with  $l(x) - l(z) = 1$ , met in the direction  $za_1a_2 \dots a_nz'$  of  $P_z^C$ . On the other hand, if  $z' \notin V_H(P_z^C)$ , there must be a vertex  $x'$  such that  $l(x') - l(z') = 1$  and  $x'$  is the first vertex with  $l(x') - l(z') = 1$ , met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ . Let us now focus on two paths  $P_{zx} = za_1a_2 \dots a_ix, i \geq 2$ , and  $P_{x'z'} = x'a_j \dots a_nz', j \geq 2$ . Let  $s$  be an arbitrary vertex of  $V_L(P_{zx})$ . If  $z \notin V_H(P_z^C)$ , we will also consider an arbitrary vertex of  $V_L(P_{x'z'})$ , denoted by  $s'$ . Now, we construct the digraph  $D$ , which meets the conditions of Lemma 3.2, as follows:

**Case 1:** Suppose that  $x'$  does not exist

Let  $w$  be the first vertex, met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ , such that  $l(z') - l(w) = l(x) - l(s)$ , and Let  $y$  be the first vertex, met in the direction of  $wa_{i'} \dots a_nz'$ , such that  $l(y) - l(w) = l(z) - l(s)$ . Note that  $y \neq z'$ . It is easy to see that  $P_{wz'} = wa_{i'} \dots a_nz'$ , and  $P_{sx} = sa_{j'} \dots a_ix$  are of type  $r = l(z') - l(w)$ , and  $P_{wy} = wa_{i'} \dots y$ , and  $P_{sz} = s \dots a_1z$  are of type  $r' = l(z) - l(s)$ . Applying Proposition 2.6, we can construct two oriented paths  $P_1$  of type  $r$  and  $P_2$  of type  $r' = r - 1$ , with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$ , which map homomorphically to  $P_{wz'}, P_{sx}$ , and  $P_{wy}, P_{sz}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $x$ , and  $z'$  ( $s$ , and  $w$ ), and  $v_1$  (respectively,  $v_2$ ) maps to  $s$  and  $w$  ( $z$ , and  $y$ ). To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $x, y, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2. To show that the condition (b) of this lemma also holds, it is enough to see that the net length of  $D$  is one, i.e.,  $l(u) - l(v) = 1$ ; however,  $l(x) - l(y) \leq 0$  in  $P_z^C$ .

**Case 2:** Suppose that  $x'$  exists and  $l(z) - l(s) \neq l(z') - l(s')$

Let  $r = l(z) - l(s)$ , and  $r' = l(z') - l(s')$  in  $P_z^C$ . First, we assume that  $r > r'$ ; hence  $r \geq (l(x') - l(s'))$ , as  $l(x') - l(s') = r' + 1$ . Let  $w$  be the first vertex, met in the direction of  $P_{zs} = za_1 \dots s$ , such that  $l(z) - l(w) = l(x') - l(s')$ , and let  $y$  be the first vertex, met in the direction of  $P_{wz} = wa_{i'} \dots a_1z$ , such that  $l(y) - l(w) = l(z') - l(s')$ . Note that  $y \neq z$ . It is easy to check that  $P_{s'x'} = s' \dots a_jx'$ , and  $P_{wz} = wa_{i'} \dots a_1z$  are of type  $r' + 1$ , and  $P_{wy} = wa_{i'} \dots y$ , and  $P_{s'z'} = s' \dots a_nz'$  are of type  $r'$ . Similar to Case 1, we can apply Proposition 2.6 to find  $P_1$  and  $P_2$  with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$  for  $P_{s'x'}, P_{wz}$ , and  $P_{s'z'}, P_{wy}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $x'$ , and  $z$  ( $s'$  and  $w$ ) and  $v_1$  (respectively,  $v_2$ ) maps to  $s'$  and  $w$  ( $z'$  and  $y$ ). To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $x', y, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2. To show that the condition (b) of this lemma also holds, it is enough to see that the net length of  $D$  is one, i.e.,  $l(u) - l(v) = 1$ ; however,  $l(x') - l(y) \geq 4$  in  $P_z^C$ .

Second, we assume that  $r < r'$ . Then, the only difference is that  $w$  and  $y$  are in  $P_{s'z'}$  rather than

$P_{sz}$ , and  $x, y$  are the representative pair of  $P_z^C$  rather than  $x', y$  in Lemma 3.2. Then  $l(x) - l(y) \leq 0$ , as  $l(z') - l(z) \geq 2$ ; hence the condition (b) of Lemma 3.2 holds as  $l(u) - l(v) = 1$ .

**Case 3:** Suppose that  $x'$  exists and  $l(z) - l(s) = l(z') - l(s')$

Let  $w \neq z$  (respectively,  $w' \neq z'$ ) be a vertex of  $P_{zx} = za_1 \dots a_ix$  (respectively,  $P_{x'z'} = x'a_j \dots a_nz'$ ) with  $l(z) = l(w)$  ( $l(w') = l(z')$ ), and let  $r = l(z) - l(s) = l(z') - l(s')$ . One can easily check that  $P_{s'z'}, P_{s'w'}, P_{sz}, P_{sw}$  are of type  $r$ . Applying Proposition 2.6, we can construct two oriented paths  $P_1$  and  $P_2$  of type  $r$ , with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$ , which map homomorphically to  $P_{s'z'}, P_{sw}$ , and  $P_{s'w'}, P_{sz}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $w$ , and  $z'$  ( $s$ , and  $s'$ ), and  $v_1$  (respectively,  $v_2$ ) maps to  $s$  and  $s'$  ( $z$ , and  $w'$ ). To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $w, w', z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2. To show that the condition (b) of this lemma also holds, it is enough to see that the net length of  $D$  is zero, i.e.,  $l(u) - l(v) = 0$ ; however,  $l(w) - l(w') \leq -2$  in  $P_z^C$ .  $\diamond$

### 3.3 Oriented Cycles $C$ with $\lambda(C) = 1$

Before we give a dichotomy for this subclass of oriented cycles with some loops, we distinguish two special vertices  $s$  and  $t$  of oriented cycles with net length one. These two vertices play an important role in our study of this subclass of oriented cycles with some loops.

**Lemma 3.7** *Let  $C$  be an oriented cycle of net length one and  $b_0$  be an arbitrary vertex of  $C$ . Let  $t$  be the first vertex of  $V_H(P_{b_0}^C)$ , met in the direction  $b_0b_1 \dots b_pb_0$ , and  $s$  be the last vertex of  $V_L(P_{b_0}^C)$ , met in the direction  $b_0b_1 \dots b_pb_0$ . Then the pair  $s, t$  in  $C$  is independent of the choice of  $b_0$ .*

**Proof:** Recall that each vertex of  $P_{b_0}^C$  has a unique counterpart in  $C$ . So, when we refer to a vertex in  $P_{b_0}^C$ , one can imagine its corresponding vertex in  $C$ . Let  $P_{b_0}^C = b_0b_1 \dots b_pb_0$  and  $P_{a_0}^C = a_0a_1 \dots a_pa_0$  be two arbitrary oriented paths starting from  $b_0$ , and  $a_0$ , respectively. For simplicity, we replace the last  $b_0$  and  $a_0$  in  $P_{b_0}^C$  and  $P_{a_0}^C$  with  $b_{p+1}$  and  $a_{p+1}$ , respectively, i.e.,  $P_{b_0}^C = b_0b_1 \dots b_pb_{p+1}$  and  $P_{a_0}^C = a_0a_1 \dots a_pa_{p+1}$ . We will show that  $t$  is the first vertex of  $V_H(P_{a_0}^C)$ , met in the direction  $a_0a_1 \dots a_pa_{p+1}$ . For  $s$  the proof is similar.

Note that both  $P_{b_0}^C$  and  $P_{a_0}^C$  traverses  $I(C)$  in the same direction by the assumption of traversing oriented cycles in the direction of positive net length. Now, the following cases may happen:

**Case 1:** Suppose  $a_0$  occurs on the oriented path from  $b_0$  to  $t$  (inclusively)

It is easy to see that no vertex on the oriented path from  $a_0$  to  $t$  has a level equal or more than  $t$  in  $P_{a_0}^C$ . Recall that  $\lambda(C) = 1$ , i.e.,  $l(b_{p+1}) - l(b_0) = 1$ . Now, we show that no vertex of  $P_{a_0}^C$  on the oriented path  $Q$  from  $t$  to  $a_{p+1}$  has a level more than  $t$ . In fact, the portion of  $Q$  which is from  $t$  to  $b_{p+1}$  does not have such a vertex. Suppose, on the other hand, that this vertex occurs in the portion of  $Q$  from  $b_{p+1}$  to  $a_{p+1}$ . This is impossible since  $l(b_{p+1}) - l(b_0) = 1$ , i.e., the vertices of this portion in  $P_{a_0}^C$  are exactly in one level more than the same vertices in the same

portion in  $P_{b_0}^C$  reaching at most to the level of  $t$  in  $P_{a_0}^C$ . (Recall that the levels of vertices of this portion in  $P_{b_0}^C$  are strictly less than the level of  $t$ .) Hence,  $t$  is also the first vertex of  $V_H(P_{a_0}^C)$ , met in the direction  $a_0 a_1 \dots a_p a_{p+1}$ .

**Case 2:** Suppose  $a_0$  occurs on the oriented path from  $t$  to  $b_{p+1}$  (inclusively)

Let  $Q$  be the portion of  $P_{a_0}^C$  from  $b_{p+1}$  to  $a_{p+1}$ . Since  $t$  is the first vertex of  $V_H(P_{b_0}^C)$ , met in the direction  $b_0 b_1 \dots b_p b_0$ , it is easy to see that  $t$  is also the first vertex of  $V_H(Q)$ , met in the direction  $b_{p+1} \dots a_p a_{p+1}$ . Now, we show that all vertices of  $P_{a_0}^C$  on the oriented path  $Q'$  from  $a_0$  to  $b_{p+1}$  have levels less than  $t$ . In fact, there might be some vertices of the same level as  $t$  on  $Q'$  when we see  $Q'$  as a portion of  $P_{b_0}^C$ . However, since  $l(b_{p+1}) - l(b_0) = 1$ , these vertices of this portion  $Q'$  in  $P_{a_0}^C$  are exactly in one level less than  $t$  in  $P_{a_0}^C$ , as  $t$  occurs on  $Q$ . Hence,  $t$  is also the first vertex of  $V_H(P_{a_0}^C)$ , met in the direction  $a_0 a_1 \dots a_p a_{p+1}$ .  $\diamond$

Let  $C$  be an oriented cycle with some loops such that  $\lambda(C) = 1$ . In this subsection, we assume that  $s$  and  $t$  are fixed vertices of  $I(C)$  introduced in Lemma 3.7. Recall that  $C_3^1$  is an oriented cycle with the vertex set  $\{1, 2, 3\}$ , and the arc set  $\{12, 23, 13\}$ .

**Theorem 3.8** *Let  $C'$  be an oriented cycle with some loops such that  $\lambda(C') = 1$ . If  $C'$  is one of the following digraphs, then  $\text{MinHOM}(C')$  is polynomial time solvable. Otherwise,  $\text{MinHOM}(C')$  is NP-hard.*

- (a) Any oriented cycle  $C'$  such that  $I(C') = C_3^1$ .
- (b) Any oriented cycle  $C'$  such that  $I(C') \neq C_3^1$ , and  $C'$  has at most two loops, which are the loops of  $s$  and  $t$  as defined earlier.

**Proof:** It is trivial to check that  $C'$  has a Min-Max ordering when  $I(C') = C_3^1$ . Thus, we assume that  $I(C') \neq C_3^1$ . To prove part (b), suppose at least one of  $s$  and  $t$  has a loop, and no other vertex of  $C'$  has a loop. We wish to obtain a Min-Max ordering  $\ll$  for  $C'$ . Let  $b_0$  be an arbitrary vertex of  $C'$ . In what follows  $l(u)$  represents the level of  $u$  in  $P_{b_0}^{C'}$ . Once, we have  $P_{b_0}^{C'}$ , we can order the vertices of  $C'$  with the following rules (note that we do not order  $b_{p+1}$ , since it a copy of  $b_0$ ):

1. If  $l(u) < l(v)$  then  $u \ll v$ ;
2. If  $l(u) = l(v)$ , and  $u$  has been met earlier than  $v$  in the direction  $b_0 b_1 \dots b_p b_{p+1}$ , then  $v \ll u$ .

Consider that  $t$  has the highest, and  $s$  has the lowest order in  $\ll$ . Thus, there is no crossing pair including arcs  $ss$  or  $tt$ . It is also easy to check that there is no crossing pair between the other arcs. Hence,  $\ll$  is a Min-Max ordering. (see Figure 3.)

Now, It remains to prove that if a vertex  $z$  of  $C'$  other than  $s$  and  $t$  has a loop then  $\text{MinHOM}(C')$  is NP-hard. To do so, it is sufficient by Lemma 3.5 to show that  $\text{MinHOM}(C)$  is NP-hard, where  $C$  is an oriented cycle obtained from  $C'$  by removing all loops but the loop of  $z$ . Now, if  $z$  is neither a source nor a sink in  $I(C)$ , then  $\text{MinHOM}(C)$  is NP-hard by Lemma 3.4. So, we assume that  $z$  is either a source or a sink in  $I(C)$ . Without loss of generality we assume that  $z$  is a sink in  $I(C)$ .

Consider the oriented path  $P_z^C = z a_1 a_2 \dots a_n z$ . For simplicity, let us show the last  $z$  by  $z'$ , i.e.,  $P_z^C = z a_1 a_2 \dots a_n z'$ . It follows from the definition of  $P_z^C$  and the net length of  $I(C)$  that  $l(z') - l(z) = 1$ .

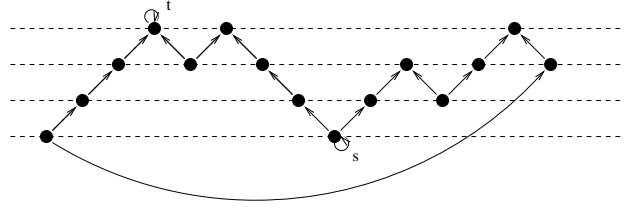


Figure 3: Dashed lines represent levels. The higher dashed lines, the higher levels. The righter vertex, the lower order.

Observe that since  $z \neq t$ , we will always have vertices  $x, q \neq z, z'$  in  $P_z^C$  such that  $l(x) - l(z) = 1$ , and  $l(q) - l(z) = 0$ . Among such vertices, we will choose  $x$  and  $q$  as the first vertices with  $l(x) - l(z) = 1$  and  $l(q) - l(z) = 0$ , met in the direction  $z a_1 a_2 \dots a_n z'$  of  $P_z^C$ . On the other hand, if  $z' \notin V_H(P_z^C)$ , there must be a vertex  $x'$  such that  $l(x') - l(z') = 1$ , otherwise  $x'$  does not exist. Among such vertices, we will choose  $x'$  as the first vertex with  $l(x') - l(z') = 1$ , met in the direction  $z' a_n \dots a_2 a_1 z$  of  $P_z^C$ . Let us now focus on two paths  $P_{zx} = z a_1 a_2 \dots a_i x$  and  $P_{x'z'} = x' a_j \dots a_n z'$ . Let  $s$  be an arbitrary vertex of  $V_L(P_{zx})$ . If  $z \notin V_H(P_z^C)$ , we will also consider an arbitrary vertex of  $V_L(P_{x'z'})$ , denoted by  $s'$ . We now construct a digraph  $D$ , which meets the conditions of Lemma 3.2:

**Case 1:** Suppose that  $x'$  does not exist

Since  $x'$  does not exist, we have  $z' \in V_H(P_z^C)$ . Let  $w$  be the first vertex, met in the direction  $z' a_n \dots a_2 a_1 z$  of  $P_z^C$ , such that  $l(z') - l(w) = l(q) - l(s)$ , and Let  $y$  be the first vertex, met in the direction of  $w a_i \dots a_n z'$ , such that  $l(y) - l(w) = l(z) - l(s)$ . Note that  $y \neq z'$ , as  $z \neq t$ . It is easy to see that  $P_{wz'} = w a_i \dots a_n z'$ ,  $P_{sq} = s a_j \dots a_i q$ ,  $P_{wy} = w a_i \dots y$ , and  $P_{sz} = s \dots a_1 z$  are all of type  $r = l(z') - l(w) = l(z) - l(s)$ . Applying Proposition 2.6, we can construct two oriented paths  $P_1$  and  $P_2$  of type  $r$ , with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$ , which map homomorphically to  $P_{wz'}$ ,  $P_{sq}$ , and  $P_{wy}$ ,  $P_{sz}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $q$ , and  $z'$  ( $s$ , and  $w$ ), and  $v_1$  (respectively,  $v_2$ ) maps to  $s$  and  $w$  ( $z$ , and  $y$ ). To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $q, y, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2. To show that the condition (b) of this lemma also holds, it is enough to see that the net length of  $D$  is zero, i.e.,  $l(u) - l(v) = 0$ ; however,  $l(q) - l(y) = -1$  in  $P_z^C$ .

**Case 2:** Suppose that  $x'$  exists and  $l(z) - l(s) \neq l(z') - l(s')$

$D$  is constructed exactly like Case 2 in the proof of Theorem 3.6. Note that  $l(u) - l(v) = 1$  in  $D$ . If  $r' < r$ ,  $l(x') - l(y) = 3$  as  $l(z') - l(z) = 1$ ; hence the condition (b) of Lemma 3.2 holds as  $l(u) - l(v) = 1$ . One the other hand, if  $r < r'$ , then  $l(x) - l(y) = 1$ , which does not necessarily guarantee that the condition (b) of Lemma 3.2 holds, as  $l(u) - l(v) = 1$ . However, Since  $l(x') - l(x) = 1$ , there is no homomorphism, mapping  $u$  to  $x$  and  $v$  to  $y$  due to existence of  $x'$ . In fact, such a homomorphism  $f$  maps  $D$  to the oriented cycle, existing between  $x$  and  $y$  in  $P_z^C$ , such that  $f(u) = x$ ,  $f(v) = y$ , and there is a vertex  $u'$  in  $D$ , for which we have  $f(u') = x'$ , since  $x'$  is in the oriented cycle between  $x$ , and  $y$ . We denote by  $l_D(u)$ , the level of

vertex  $u$  in  $D$ . We can easily see that  $l_D(u') > l_D(u)$ , as  $l(x') > l(x)$ . This is a contradiction, since both  $P_1$  and  $P_2$  are of type  $r+1$  and  $r$  and no vertex of  $D$  has a level more than  $u$ .

**Case 3:** Suppose that  $x'$  exists and  $l(z) - l(s) = l(z') - l(s')$

$D$  is constructed exactly like Case 3 in the proof of Theorem 3.6. To show that the condition (b) of Lemma 3.2 also holds, it is enough to see that  $l(u) - l(v) = 0$ ; however,  $l(u) - l(u') = -1$  in  $P_z^C$ .  $\diamond$

### 3.4 Oriented Cycles $C$ with $\lambda(C) = 0$

We begin this subsection with introducing two special pairs of vertices  $s_1, s_2$  and  $t_1, t_2$  for oriented cycles  $C$  of the form  $l^+h^+$ . Let  $C$  be a balanced oriented cycle of the form  $l^+h^+$ , and let  $b_0$  and  $P_{b_0}^C$  be the absolute base and absolute base path of  $C$ , respectively. (note that for each balanced oriented cycle  $C$  of the form  $l^+h^+$ , the absolute base is unique.) Let  $t_1$  be the first vertex and  $t_2$  be the last vertex of  $V_H(P_{b_0}^C)$ , met in the direction  $b_0b_1 \dots b_p b_0$ , and  $s_1$  be the first vertex and  $s_2$  be the last vertex of  $V_L(P_{b_0}^C)$ , met in the direction  $b_0b_p \dots b_1$ . It is easy to see that all these four vertices are fixed in  $C$  and  $b_0 = s_1$ , as  $C$  is balanced of the form  $l^+h^+$ . Note that  $t_1, t_2$  (respectively,  $s_1, s_2$ ) are not necessarily distinct; we can assume a case where  $|V_H(P_{b_0}^C)| = 1$ . (respectively,  $|V_L(P_{b_0}^C)| = 1$ .) For an oriented cycle with some loops  $C$ ,  $s_1, s_2, t_1, t_2$  are defined as  $s_1, s_2, t_1, t_2$  in  $I(C)$ , respectively. Recall that  $C_4^0$  is an oriented cycle with the vertex set  $C_4^0 = \{1, 2, 3, 4\}$ , and the arc set  $\{12, 32, 14, 34\}$ .

**Theorem 3.9** *Let  $C'$  be an oriented cycle with some loops such that  $I(C')$  is balanced. If  $C'$  is one of the following digraphs, then  $\text{MinHOM}(C')$  is polynomial time solvable. Otherwise,  $\text{MinHOM}(C')$  is NP-hard.*

- (a) Any oriented cycle  $C'$  such that  $I(C') = C_4^0$ , and  $C'$  has at most two loops.
- (b) Any oriented cycle  $C'$  such that  $I(C')$  is of the form  $l^+h^+$ , and  $C'$  has at most two loops, which are the loops of either  $s_1$  and  $t_2$  or  $s_2$  and  $t_1$  as defined earlier.

**Proof:** If  $I(C')$  is not of the form  $l^+h^+$  and  $I(C') \neq C_4^0$ , then  $\text{MinHOM}(I(C'))$  is NP-hard; hence by Lemma 3.5,  $\text{MinHOM}(C')$  is NP-hard. It is trivial to check that  $C'$  has a Min-Max ordering, when  $I(C') = C_4^0$ , and  $C'$  has at most two loops. If  $I(C') = C_4^0$ , and  $C'$  has three or four loops, then  $B(C')$  has a binet as an induced subgraph and  $\text{MinHOM}(C')$  is NP-hard by Proposition 2.8.

Now, suppose that  $C'$  is an oriented cycle with some loops such that  $I(C') \neq C_4^0$  is of the form  $l^+h^+$ , and  $C'$  has at most two loops, which belong to either  $s_1$  and  $t_2$  or  $s_2$  and  $t_1$ . Without loss of generality, assume that at least one of  $s_2$  and  $t_1$  has a loop and no other vertex of  $C'$  has a loop. (see Figure 4.) We split the oriented cycle  $C'$  to two oriented paths  $P_1$ , and  $P_2$  from  $s_1$  to  $s_2$ . In what follows  $l_{P_1}(u)$  (respectively,  $l_{P_2}(u)$ ) represents the level of  $u$  in  $P_1$  (respectively,  $P_2$ ), where  $l_{P_1}(s_1) = 0$ , and  $l_{P_2}(s_1) = 0$ . Since  $I(C')$  is of the form  $l^+h^+$  and  $I(C') \neq C_4^0$ , one of  $P_1$  or  $P_2$ , say  $P_1$ , contains all vertices of  $V_H(P_{s_1}^C) = V_H(C')$ , and  $P_2$  contains all vertices of  $V_L(P_{s_1}^C) = V_L(C')$ . Hence,  $l_{P_2}(u) < l_{P_1}(t_1)$  for all  $u \in V(P_2)$ . We wish to obtain a Min-Max ordering  $\ll$  for  $C'$ . We can order the vertices of  $C'$  with the following rules:

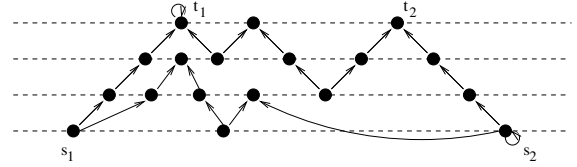


Figure 4: Dashed lines represent levels. The higher dashed lines, the higher levels. The righter vertex, the lower order.

1. If  $u \in P_i, i = 1, 2, v \in P_j, j = 1, 2$ , and  $l_{P_i}(u) < l_{P_j}(v)$  then  $u \ll v$ ;
2. If  $u, v \in P_i, i = 1, 2$ , and  $l_{P_i}(u) = l_{P_i}(v)$ , and  $u$  has been met earlier than  $v$  in the direction  $s_1 \dots s_2$  in  $P_i$ , then  $v \ll u$ .
3. If  $u \in P_1, v \in P_2$ , and  $l_{P_1}(u) = l_{P_2}(v)$ , then
  - 3.1. if  $u$  is in the oriented path between  $s_1$  and  $t_1$  in  $P_1$ , then  $v \ll u$ ;
  - 3.2. otherwise,  $u \ll v$ .

Note that  $t_1$  has the highest and  $s_2$  has the lowest order in  $\ll$ . Thus, there is no crossing pair including arcs  $s_2s_2$  or  $t_1t_1$ . It is also easy to check that there is no crossing pair between the other arcs since  $I(C') \neq C_4^0$  is of the form  $l^+h^+$ , and  $l_{P_2}(u) < l_{P_1}(t_1)$  for all  $u \in V(P_2)$ ; hence,  $\ll$  is a Min-Max ordering. (see Figure 4.)

It remains to prove that  $\text{MinHOM}(C')$  is NP-hard for all oriented cycles  $C'$  with some loops, where  $I(C')$  is of the form  $l^+h^+$ , and  $C'$  does not fulfill the conditions of part (b). Let  $b_0$  be the absolute base of  $C'$ . The following lemmas cover this fact.  $\diamond$

**Lemma 3.10** *Let  $C'$  be an oriented cycle with some loops such that  $I(C')$  is balanced and of the form  $l^+h^+$ . If a vertex  $z$  of  $C'$ , which is neither in  $V_H(P_{b_0}^C)$  nor in  $V_L(P_{b_0}^C)$ , has a loop, then  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** It is sufficient by Lemma 3.5 to show that  $\text{MinHOM}(C)$  is NP-hard, where  $C$  is an oriented cycle obtained from  $C'$  by removing all loops but the loop of  $z$ . Now, if  $z$  is neither a source nor a sink in  $I(C)$ , then  $\text{MinHOM}(C)$  is NP-hard by Lemma 3.4. So, we assume that  $z$  is either a source or a sink in  $I(C)$ . Without loss of generality, we assume that  $z$  is a sink in  $I(C)$ .

Consider the oriented path  $P_z^C = za_1a_2 \dots a_nz$ . For simplicity, let us show the last  $z$  by  $z'$ , i.e.,  $P_z^C = za_1a_2 \dots a_nz'$ . Since  $z$  is neither in  $V_H(P_{b_0}^C)$  nor in  $V_L(P_{b_0}^C)$ , we will always have a unique vertex  $x \neq z, z'$  in  $P_z^C$  such that  $l(x) - l(z) = 1$ , and  $x$  is the first vertex with  $l(x) - l(z) = 1$ , met in the direction  $za_1a_2 \dots a_nz'$  of  $P_z^C$ . On the other hand, there is also a unique vertex  $x' \neq z'$  such that  $l(x') - l(z') = 1$ , and  $x'$  is the first vertex with  $l(x') - l(z') = 1$ , met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ . Let us now focus on two paths  $P_{zx} = za_1a_2 \dots a_ix$  and  $P_{x'z'} = x'a_j \dots a_nz'$ . Let  $s$  (respectively,  $s'$ ) be an arbitrary vertex of  $V_L(P_{zx})$  (respectively,  $V_L(P_{x'z'})$ ). The following cases may happen:

**Case 1:** Suppose that  $l(z) - l(s) \neq l(z') - l(s')$   
 $D$  is constructed exactly like Case 2 in the proof of Theorem 3.6. Note that  $l(u) - l(v) = 1$  in  $D$ . If  $r' < r$  (respectively,  $r < r'$ ),  $l(x') - l(y) = 2$  (respectively,  $l(x) - l(y) = 2$ ) as  $l(z') - l(z) = 0$ ; hence the condition (b) of Lemma 3.4 holds since  $l(u) - l(v) = 1$ .

**Case 2:** Suppose that  $l(z) - l(s) = l(z') - l(s')$

$D$  is constructed exactly like Case 3 in the proof of Theorem 3.6. Since  $l(u) - l(v) = 0$  and  $l(w) - l(w') = 0$ , the condition (b) of Lemma 3.2 is not easily implied. However, due to the existence of  $x$ , this condition also holds. In fact, if a homomorphism  $f$  of  $D$  to  $C$  exists such that  $f(u) = w, f(v) = w'$ , it must map the vertices of  $D$  to the oriented cycle between  $w$  and  $w'$  in  $P_z^C$ . Thus, there is a vertex  $u'$  in  $D$ , for which we have  $f(u') = x$ , as  $x$  is in the oriented cycle between  $w$  and  $w'$ . We denote by  $l_D(u)$ , the level of vertex  $u$  in  $D$ . It is easy to see that  $l_D(u') > l_D(u)$ , as  $l(x) > l(w)$ . This is a contradiction since both  $P_1$  and  $P_2$  are of type  $r$  and no vertex of  $D$  has a level more than  $u$ .  $\diamond$

**Lemma 3.11** *Let  $C'$  be an oriented cycle with some loops such that  $I(C')$  is balanced and of the form  $l^+h^+$ . If a vertex  $z$  of  $V_H(P_{b_0}^{C'})$  (respectively,  $V_L(P_{b_0}^{C'})$ ), which is neither  $t_1$ , nor  $t_2$  (respectively, neither  $s_1$  nor  $s_2$ ) has a loop, then  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** Without loss of generality, assume that  $z \in V_H(P_{b_0}^{C'})$ , and clearly it is a sink. Similar to the proof of Lemma 3.10, we consider  $C$ , which is an oriented cycle obtained from  $C'$  by removing all loops but the loop of  $z$ .

Consider the oriented path  $P_z^C = za_1a_2 \dots a_nz$ . For simplicity, let us show the last  $z$  by  $z'$ , i.e.,  $P_z^C = za_1a_2 \dots a_nz'$ . Since  $|V_H(P_{b_0}^C)| \geq 3$ , we will always have a vertex  $x \neq z, z'$  in  $P_z^C$  such that  $l(x) - l(z) = 0$ , and  $x$  is the first vertex with  $l(x) - l(z) = 0$ , met in the direction  $za_1a_2 \dots a_nz'$  of  $P_z^C$ . On the other hand, there is another vertex  $y \neq z, z'$  such that  $l(y) - l(z') = 0$ , and  $y$  is the first vertex with  $l(y) - l(z') = 0$ , met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ . Note that  $x$  and  $y$  are distinct vertices, as  $|V_H(P_{b_0}^C)| \geq 3$ , and both of them are in  $V_H(P_{b_0}^C)$ , as  $z \in V_H(P_{b_0}^C)$ . Let us now focus on two paths  $P_{zx} = za_1a_2 \dots a_ix$  and  $P_{yz'} = ya_j \dots a_nz'$ . Let  $s$  (respectively,  $s'$ ) be an arbitrary vertex of  $V_L(P_{zx})$  (respectively,  $V_L(P_{yz'})$ ). Without loss of generality, assume that  $l(z') - l(s') \leq l(s) - l(z)$ . Observe that neither  $s$  nor  $s'$  is not in  $V_L(P_z^C)$ , as  $I(C)$  is of the form  $l^+h^+$ , and  $z \neq t_1, t_2$ . In other words, there exists a vertex  $s'' \in V_L(P_z^C)$ , which is in the oriented path  $P_{xy} = x \dots y$  of  $P_z^C$ .

Let  $w$  be the first vertex, met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ , such that  $l(z') - l(w) = l(x) - l(s)$ . It is easy to see that  $P_{wz'} = wa_{i'} \dots a_nz'$ , and  $P_{sx} = sa_{j'} \dots a_ix$  are of type  $r = l(z') - l(w)$ , and  $P_{wy} = wa_{i'} \dots y$ , and  $P_{sz} = s \dots a_1z$  are of type  $r' = l(z) - l(s) = r$ . Applying Proposition 2.6, we can construct two oriented paths  $P_1$ , and  $P_2$  of type  $r$ , with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$ , which map homomorphically to  $P_{wz'}, P_{sx}$ , and  $P_{wy}, P_{sz}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $x$ , and  $z'$  ( $s$ , and  $w$ ), and  $v_1$  (respectively,  $v_2$ ) maps to  $s$  and  $w$  ( $z$ , and  $y$ ). To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $x, y, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2.

Note that  $l(u) - l(v) = 0$  in  $D$ . Since  $l(x) - l(y) = 0$ , the condition (b) of Lemma 3.2 is not easily implied, as  $l(u) - l(v) = 0$ . However, there is no homomorphism, mapping  $u$  to  $x$  and  $v$  to  $y$  due to existence of  $s''$ . In fact, such a homomorphism  $f$  maps  $D$  to the oriented path, existing between  $x$  and  $y$  in  $P_z^C$ ,

such that  $f(u) = x, f(v) = y$ , and there is a vertex  $u'$  in  $D$ , for which we have  $f(u') = s''$  as  $s''$  is in the oriented cycle between  $x$ , and  $y$ . This way, we must have:  $l_D(u) - l_D(u') = l(x) - l(s'') = l(z') - l(s'')$ , which is a contradiction since  $D$  does not contain  $u'$  with such a level.  $\diamond$

**Lemma 3.12** *Let  $C'$  be an oriented cycle with some loops such that  $I(C')$  is balanced and of the form  $l^+h^+$ . If distinct vertices  $t_1$  and  $t_2$  (respectively,  $s_1$  and  $s_2$ ) have loops, then  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** Without loss of generality, we prove this Lemma for  $t_1$  and  $t_2$ , and we consider  $C$ , which is an oriented cycle obtained from  $C'$  by removing all loops but the loops of  $t_1$ , and  $t_2$ .

Let  $z = t_1$  and  $x = t_2$ . Consider the oriented path  $P_z^C = za_1a_2 \dots a_nz'$ . Let  $s$  be an arbitrary vertex of  $V_L(P_{zx})$ , where  $P_{zx} = za_1a_2 \dots a_ix$ . Observe that  $s \notin V_L(P_z^C)$  since  $I(C)$  is of the form  $l^+h^+$ . Hence, there exists a vertex  $s'$ , which is in the oriented path  $P_{xz'} = xa_{i+1} \dots a_nz'$  of  $P_z^C$ , where  $l(s) - l(s') = 1$ . Among such vertices, we will choose  $s'$  as the first vertex with  $l(s) - l(s') = 1$ , met in the direction  $z'a_n \dots a_2a_1z$  of  $P_z^C$ . Let  $y$  be the first vertex, met in the direction of  $P_{s'z'} = s'a_{i'} \dots a_nz'$ , such that  $l(y) - l(s') = l(z) - l(s)$ . Note that  $y \neq z'$ . We will virtually assume a vertex  $x'$  such that  $x$  dominates this vertex. (note that  $x = t_2$  has a loop in  $C$ .) It is easy to see that  $P_{s'z'} = s'a_{i'} \dots a_nz'$ , and  $P_{sx'} = sa_{j'} \dots a_ixx'$  are of type  $r = l(z') - l(s')$ , and  $P_{s'y} = s'a_{i'} \dots y$ , and  $P_{sz} = s \dots a_1z$  are of type  $r' = r - 1$ .

Applying Proposition 2.6, we can construct two oriented paths  $P_1$  of type  $r$  and  $P_2$  of type  $r'$ , with terminal vertices  $u_1, v_1$ , and  $u_2, v_2$ , which map homomorphically to  $P_{s'z'}, P_{sx'}$ , and  $P_{s'y}, P_{sz}$ , respectively, such that  $u_1$  (respectively,  $u_2$ ) maps to  $x'$ , and  $z'$  ( $s$ , and  $s'$ ), and  $v_1$  (respectively,  $v_2$ ) maps to  $s$  and  $s'$  ( $z$ , and  $y$ ). One can easily see that all vertices that these homomorphisms map to  $x'$ , can also be mapped to  $x$ , since  $x$  has a loop in  $C$ . Now, we construct a digraph  $D$ , which fulfills the conditions of Lemma 3.3. To construct  $D$ , we will join these two oriented paths at the vertex  $v_1$  of  $P_1$ , and the vertex  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . Let all  $c_i(u) = 0$  apart from  $c_i(u) = +\infty, i \in V(P_{xs'}) - \{x, s'\}, u \in V(D)$ , where  $P_{xs'} = xa_{i+1} \dots s'$ .

One can easily check that  $u, v$  of  $D$  and  $x, y, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (b), (d), and (e) of Lemma 3.3. To show that the condition (c) of this lemma also holds, it is enough to see that there is no homomorphism of  $D$  to  $C$ , which maps  $u$  to  $x$ , and  $v$  to  $y$ , unless one of the vertices of  $D$  maps to a vertex  $i \in V(P_{xs'}) - \{x, s'\}$ , i.e., the cost of homomorphism is infinity, meeting the condition (c) of Lemma 3.3. Thus,  $\text{MinHOM}(C)$  is NP-hard.  $\diamond$

**Lemma 3.13** *Let  $C'$  be an oriented cycle with some loops such that  $I(C')$  is balanced and of the form  $l^+h^+$ . If  $s_1$  and  $t_1$  (or  $s_2$  and  $t_2$ ) have loops, and  $|V_H(P_{b_0}^{C'})| \geq 2, |V_L(P_{b_0}^{C'})| \geq 2$ , then  $\text{MinHOM}(C')$  is NP-hard.*

**Proof:** Without loss of generality, we prove this Lemma for  $s_1, t_1$ , and we consider  $C$ , which is an oriented cycle obtained from  $C'$  by removing all loops but the loops of  $s_1$ , and  $t_1$ .

Let  $z = t_1$ . Consider the oriented path  $P_z^C = za_1a_2 \dots a_nz'$ . Since  $|V_H(P_{b_0}^C)| \geq 2$ , we will always have a vertex  $x \neq z, z'$  in  $P_z^C$  such that  $l(x) - l(z) = 0$ .

Among such vertices, we will choose  $x$  as the last vertex with  $l(x) - l(z) = 0$ , met in the direction  $za_1a_2 \dots a_nz'$  of  $P_z^C$ . Let  $s'$  be an arbitrary vertex of  $V_L(P_{zx})$ , where  $P_{zx} = za_1a_2 \dots a_ix$ . Observe that  $s' \notin V_L(P_z^C)$ , as  $I(C)$  is of the form  $l^+h^+$ . Thus, all vertices of  $V_L(P_z^C)$  are in the oriented path  $P_{xz'} = xa_{i+1} \dots a_nz'$ . One can easily see that  $s_1$  is the last vertex of  $V_L(P_z^C)$ , met in the direction  $za_1a_2 \dots a_pz'$ , since  $s_1$  is the first vertex of  $V_L(P_{b_0}^C)$ , met in the direction  $b_0b_p \dots b_1$  of  $P_{b_0}^C$ . Let  $s''$  be the first vertex of  $V_L(P_z^C)$ , met in the direction  $za_1a_2 \dots a_pz$ . It is easy to see that  $s''$  and  $s_1$  are distinct as  $|V_L(P_z^C)| = |V_L(P_{b_1}^C)| \geq 2$ .

It is easy to see that  $P_{s''x} = s'' \dots a_{i+1}x$ , and  $P_{s_1z'} = s_1a_{j'} \dots a_pz'$  are of type  $r = l(z') - l(s_1)$ .

Applying Proposition 2.6, we can construct an oriented path  $P_1$  of type  $r$  with terminal vertex  $u_1, v_1$ , which map homomorphically to  $P_{s''x}, P_{s_1z'}$ , such that  $u_1$  maps to  $x$ , and  $z'$ , and  $v_1$  maps to  $s''$  and  $s_1$ . Let  $P_2$  be an oriented path with terminal vertices  $u_2$ , and  $v_2$ , isomorphic to  $P_{zs''} = za_1 \dots s''$ , where  $u_2$  maps to  $s''$ , and  $v_2$  maps to  $z$  with this isomorphism. To construct  $D$ , we will join these two oriented paths at vertex  $v_1$  of  $P_1$ , and  $u_2$  of  $P_2$ . Let  $u = u_1$  and  $v = v_2$ . One can easily check that  $u, v$  of  $D$  and  $x, s_1, z, z'$  of  $P_z^C$ , which have unique counterparts in  $C$ , meet the conditions (a), (c), and (d) of Lemma 3.2. Note that when  $f(v_1) = s_1$ , then all vertices of  $P_2$  can map to  $s_1$ .

Observe that  $l(u) = l(v)$  in  $D$ , and there is a vertex  $u' \neq u, v$  in  $D$ , for which  $l(u') = l(u) = l(v)$ . However, the oriented path between  $x$  and  $s_1$  does not contain any vertex  $w$  with the same level as  $x$ , since  $x$  was the last vertex with the highest level, met in the direction  $za_1a_2 \dots a_nz'$  of  $P_z^C$ . Thus, there is no homomorphism of  $D$  to  $C$ , mapping  $u$  to  $x$ , and  $s_1$  to  $v$ , meeting the condition (b) of Lemma 3.2.  $\diamond$

## 4 Oriented Graphs

Our new dichotomy for oriented cycles with some loops is an important step towards a MinHOM dichotomy for oriented graphs with some loops. Recall that oriented graphs do not have  $\vec{C}_2$  as an induced subgraph. Thus, oriented graphs with some loops do not have  $C_2^2$  as an induced subgraph. On the other hand,  $\text{MinHOM}(C)$  is NP-hard for all oriented cycles with some loops  $C$ , with  $\lambda(C) \geq 2$ , except that  $C = C_2^2$  by Theorem 3.6. Hence, if an oriented graph  $H$  contains an oriented cycle with some loops  $C$ , with  $\lambda(C) \geq 2$ , as an induced subgraph, then  $\text{MinHOM}(H)$  is NP-hard by Proposition 2.4 and Theorem 3.6. We show that this fact will also hold when an oriented graph with some loops  $H$  contains an oriented cycle with possible loops, with  $\lambda(C) \geq 2$ , as an induced subgraph.

**Theorem 4.1** *Let  $H$  be an oriented graph with some loops. If  $H$  contains an oriented cycle with possible loops  $C$ , with  $\lambda(C) \geq 2$ , as an induced subgraph, then  $\text{MinHOM}(H)$  is NP-hard.*

Let  $H$  be an oriented graph with some loops such that  $H$  contains an oriented cycle with possible loops  $C$ , with  $\lambda(C) \geq 2$ . Then  $H$  must contain at least one of the digraphs  $H_1, H_2, H_3$ , shown in Figure 5, as an induced subgraph. Thus, to show that  $\text{MinHOM}(H)$  is NP-hard, it is sufficient by Proposition 2.4 to show that  $\text{MinHOM}(H_i)$ ,  $i = 1, 2, 3$  is NP-hard. As depicted in Figure 5,  $H_1$  is an oriented cycle with some loops, whereas  $H_2$  and  $H_3$  are digraphs, having an irreflexive oriented cycle  $C$  at the center, where the net

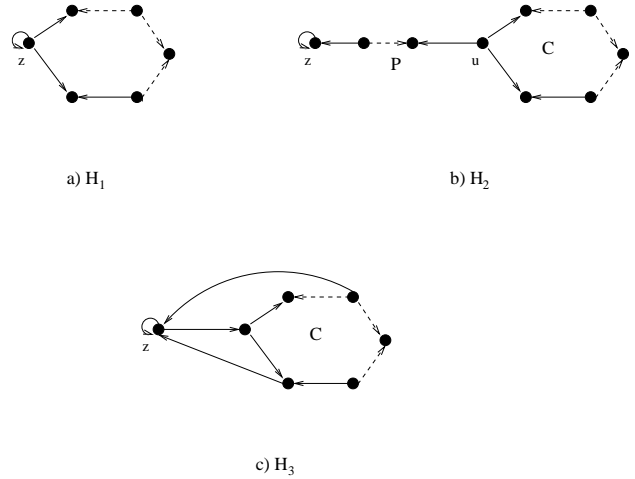


Figure 5:  $H_i$ ,  $i = 1, 2, 3$ .

length of  $C$  is at least two. Moreover,  $H_2$  has a vertex  $z$  with a loop, and an oriented path between  $z$  and a vertex  $u$  of  $C$ , and  $H_3$  has a vertex  $z$  with a loop, and at least two arcs between  $z$  and some vertices of  $C$ . We have shown in Theorem 3.6 that  $\text{MinHOM}(H_1)$  is NP-hard. We think that the same approach used to prove Theorem 3.6, can be applied to show that  $\text{MinHOM}(H_i)$ ,  $i = 2, 3$  is NP-hard. However, we do not go through the details of these two cases, since it is beyond the scope of this paper. We conclude that if Conjecture 4.1 holds, one should only seek to a dichotomy for oriented graphs having no oriented cycles or having oriented cycles with possible loops  $C$ , with  $\lambda(C) \leq 1$ .

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