Syntactic Conditions for Invertibility in Sequent Calculi

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Abstract

Formalised proofs of Cut admissibility sometimes rely on the invertibility of the rules of a sequent calculus. We present sufficient conditions for when a rule is invertible with respect to a calculus, which is important for guiding proof search. We illustrate the conditions with examples. It must be noted we give purely *syntactic* criteria; no guarantees are given as to the suitability of the rules. We also formalise some of the results in the proof assistant *Isabelle*, as a means to automating Cut admissibility proofs.

1 Introduction

Several papers (Ciabattoni & Terui 2006*b*), (Ciabattoni & Terui 2006*a*), (Rasga 2007), (Restall 1999) have sought to give syntactic or semantic conditions for a calculus to ensure that it admits Cut. In this paper, we present some easily checkable conditions which ensure that a rule is invertible with respect to the calculus in which it is defined. Note that some proofs of Cut admissibility rely on the invertibility of the rules for the calculus, such as the proofs given in (Dragalin 1988).

These results are approximately formalised in the proof assistant *Isabelle* (Nipkow et al. 2005), so that any future formalised proofs requiring the invertibility of rules can use our conditions to reduce the length, and complexity, of the proof. The first-order results are formalised using the package *Nominal Isabelle* (Tasson & Urban 2005), in which it is easy to reason about binding issues.

We build on (Dawson 2008), notably the idea that a rule in a (context-sharing) sequent calculus can be decomposed into two distinct parts. We show that the lemmata in (Dawson 2008) are logical consequences of our lemmata.

1.1 Structure of the document

We introduce some definitions in §2 which help us define, rather abstractly, a sequent calculus and some additional notions which we require. In §3, we discuss the admissibility of weakening in a calculus and prove some results about this. In §4 we give a full account of the sufficient conditions for a premiss to be derivable from the conclusion for a rule of a specific type of multisuccedent calculus, for both right rules (§4.1) and left rules (§4.2). We derive, as special cases, the conditions for invertibility in a single succedent calculus in §5. In §6 we show how the conditions in (Dawson 2008) are consequences of our conditions, and also consider a wider range of propositional calculi. In §7, we discuss first-order logics, and in §8, we discuss modal logics, as well as other directions one could take in this area. Finally, in §9, we give a brief overview of the results of the previous sections as formalised in *Isabelle*, with some examples.

2 Definitions

We distinguish between formulae and metaformulae. Let P, the set of propositional atoms, be defined by the grammar $p, q, \ldots$. Then, a *formula* is defined by the grammar:

$$A ::= P \mid \bot \mid F(A \text{ list})$$

where F ranges over constructors. For instance, if the language included conjunction, an example of a formula could be:

$$(p \wedge q) \wedge \bot \equiv \text{Conj } [\text{Conj } [p, q], \bot]$$

A formula is an expression in the object language.

Suppose that $A, B, C \ldots$ are formula variables, and $P, Q, R, \ldots$ are atom variables, then *metaformulae* are given by the following grammar:

$$\phi ::= A \mid P \mid \bot \mid F(\phi \text{ list})$$

An example of a metaformula is $\bot \supset (A \supset P)$, if implication was one of the constructors. We instantiate metaformulae with formulae.

$\Gamma, \Delta, \ldots$ are *metamultisets of formulae.* In other words, $\Gamma, \Delta, \ldots$ range over multisets of formulae.

A *sequent* is represented by $\Gamma \Rightarrow \Delta$. When we add a single (meta)formula to a (meta)multiset, we use the notation $\Gamma \oplus A$, and adding two (meta)multisets is represented by $\Gamma + \Delta$, but if the sequent is displayed in a tree, we will use the more standard "comma" notation. For instance, $\Gamma + \Gamma' \Rightarrow \Delta \oplus A$ would be displayed as:

$$\Gamma, \Gamma' \Rightarrow \Delta, A$$

The use of multisets, instead of lists or sequences, means that the results are applicable for systems in which the structural exchange rule is admissible.

In the informal analysis, we will represent a compound formula as $\star_s(\vec{A})$ or $\circ_t(\vec{B})$, where $\star_s$ is an s-ary connective, and $\vec{A}$ has length s, for example. The usual propositional formulae can be displayed in this manner, for example $A \wedge B$ would be displayed as $\text{Conj}_2\ [A, B]$.

We distinguish between *rules* and instances of rules, which we call *inferences*. Most of our lemmata are about rules, and one proves these by showing the result for arbitrary instances of the rule. Inferences can contain *no* metaformulae; they have all been instantiated with formulae. Likewise every metamultiset is instantiated, however we do not highlight the distinction between the metamultisets and their instantiations, for the sake of readability. We denote rules as $R, S, T, \ldots$ and inferences as $r, s, t \ldots$.

We impose conditions upon a calculus which ensure that the rules are invertible. To properly define these conditions, we require the following definitions about formulae:

Definition 1 (Subformulae, Metasubformulae) *The subformulae of a formula A are defined by induction on the structure of A as follows:*

- *A is a subformula of A*
- *If $F[A_1, \ldots, A_n]$ is a subformula of A, then so are $A_1, \ldots, A_n$*

The metasubformulae of a metaformula ϕ are defined by induction on the structure of ϕ as follows:

- *ϕ is a metasubformula of ϕ*
- *If $F[\phi_1, \ldots, \phi_n]$ is a subformula of ϕ, then so are $\phi_1, \ldots, \phi_n$*

$\dashv$

Context sharing logical rules (Troelstra & Schwichtenberg 2000) are such that every premiss of a rule has the same context. Each context sharing rule can, therefore, be split into two components. First, there is the *active* part of a rule: those metaformulae, and metamultisets of formulae, which cannot be arbitrarily instantiated in an inference, and the *passive* part of a rule, which can be arbitrarily instantiated. The latter part is really the context of the rule, in propositional calculi. We make this more precise:

Definition 2 (Active and Passive Metaformulae) *A metaformula ϕ is **active** for a rule R iff:*

- *ϕ cannot be arbitrarily instantiated in an instance of R, OR*
- *ϕ is a subformula occurrence of an active formula for R.*

*A metaformula is **passive** for a rule R if it is not active for R.* $\dashv$

The active part of a rule is then obtained by deleting all passive metaformulae, and metamultisets of passive formulae, from the rule. Similarly, the passive part of a rule is obtained by deleting all active metaformulae from the rule.

As an example, consider the rule from **G3cp** for $L\supset$ (Troelstra & Schwichtenberg 2000):

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \supset \psi \Rightarrow \Delta}$$

Here, the active part of the rule is:

$$\frac{\Rightarrow \phi \quad \psi \Rightarrow}{\phi \supset \psi \Rightarrow}$$

whereas the passive part consists of Γ and Δ. In more complicated rules, the active part of the rule is larger. For instance, consider the rule for $L0\supset$, one of several rules for implication on the left, from **G4ip** (Dyckhoff 1992):

$$\frac{\Gamma, P, \phi \Rightarrow \psi}{\Gamma, P, P \supset \phi \Rightarrow \psi}$$

where P is an atom variable. We have here that the active part of the rule is:

$$\frac{P, \phi \Rightarrow}{P, P \supset \phi \Rightarrow}$$

As a another example, calculi for modal logics often have *context dependent rules*. Here, the active part is yet larger, since we cannot arbitrarily instantiate a boxed multiset, for example, with a multiset in which a diamond formula occurs: the active part of:

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Box\Gamma, \Gamma' \Rightarrow \Box\phi, \Diamond\Delta, \Delta'}\ R\Box$$

is:

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box\phi, \Diamond\Delta}$$

and the passive part occurs *only in the conclusion* of the rule.

Note that structural rules have *no active metaformulae* according to this definition. As an example, take the contraction rule:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}$$

Here, ϕ can be arbitrarily instantiated, and so is *passive* by our definition.

We focus on a particular kind of rule, one in which certain structural rules (which, in general, are harmful to invertibility if primitive in the calculus), are not allowed.

Definition 3 (Decomposable rule) *We call a rule R **decomposable** iff, after deleting all active metaformulae from R to obtain R', we have:*

1. *All premisses of R' are identical AND*
2. *The antecedent (succedent) of each premiss of R' is a submultiset of the antecedent (succedent) of the conclusion of R'.*

*If every rule in a calculus $\mathcal{R}$ is decomposable, then we call $\mathcal{R}$ **decomposable**.* $\dashv$

As an example, the Contraction rule given above is *not decomposable*. For, when we remove the active formulae (of which there are none), we do not have the antecedent of the premiss being a submultiset of the antecedent of the conclusion. However, rules which have implicit weakening, such as $R\Box$ above, can be decomposable. These conditions may be verified by eye, checking the rules. We now distinguish between two kinds of decomposable rules.

Definition 4 (Normal, Implicit-Weakening Rules) *A rule R is called **normal** iff after deleting all active metaformulae from R to obtain R', the conclusion of R' is the same as each premiss of R'.*

*A rule R is called an **implicit-weakening** rule (IW rule) iff after deleting all active metaformulae from R to obtain R', the conclusion of R' is not equal to each premiss of R'.* $\dashv$

This definition of *active part* is similar to that in (Galmiche & Perrier 1994), and the definition of a passive metaformula is similar to that of a *parametric formula* from (Restall 1999). We also have similarities between the definition of a decomposable rule, and the *parameter conditions* of (Restall 1999). Here, the IW rules would violate the *non-proliferation of parametric formulae* condition, because we can have more passive metaformulae occurring in the conclusion of a rule than in the premisses of a rule. However, normal rules will satisfy the parameter conditions. Normal rules obey the regularity conditions of (Restall 1999). Note that, in the decomposition of a rule, neither part is (necessarily) a rule in its own right. We call the rule an *extension* of the active part.

In the first instance, we study two families of decomposable rules: *axioms* and *monoprincipal rules*. Axioms have no premisses, and both the antecedent and the succedent of the conclusion must contain some atom variable P, or the antecedent must contain $\bot$. In other words, the active parts must be of the form:

$$\overline{P \Rightarrow P} \qquad \overline{\bot \Rightarrow}$$

Here our approach differs from that in (Dawson 2008). There, Dawson has that *any* formula can be part of an identity sequent, whereas we restrict ourselves to atomic formulae. Most calculi are sufficiently balanced so that a general axiom can be proved admissible (i.e. $\phi \Rightarrow \phi$ for any formula ϕ), however not all calculi have this property. For instance, we cannot show this property for the sequent calculus **GL** in (Negri 2005). This restriction to atom variables in axioms, whilst more desirable from a theoretical view, has the drawback that we have to partially specify formulae.

A monoprincipal rule must have a compound formula in the conclusion of its active part, and furthermore this compound formula must be the *only* formula in the conclusion. This restriction does not allow $L0\supset$, for instance, to be a monoprincipal rule; the conclusion of the active part was:

$$P, P \supset \phi \Rightarrow$$

A sequent calculus is thus defined by some set of monoprincipal rules, joined with the set of axioms. We will usually just talk about the monoprincipal rules defining a calculus, with the understanding that the calculus contains the set of axioms.

The decomposition makes it straightforward to identify the *principal metaformula of a rule*; it is any metaformula which appears in the conclusion of the active part of the rule. In the example given earlier of $L\supset$ from **G3cp**, we can clearly see that $\phi \supset \psi$ is principal for the rule. For monoprincipal rules, we can also define *principal on the left* and *principal on the right* for a rule by noting whether the single metaformula in the conclusion of the active part of the rule is in the antecedent or succedent, respectively. The restriction to monoprincipal rules is akin to the restriction to *single principal constituents* from (Restall 1999). Note that the only condition, then, that a set of normal monoprincipal rules does not fulfil from (Restall 1999) is that of *matching of principal constituents*. Thus, we can also eliminate non-principal cuts from such a calculus.

The *principal formula of an inference* is that formula which instantiates the principal metaformula of a rule. Similarly, the *active part of an inference* is the instantiation of the active part of a rule.

We need the notion of *derivability*, and, because we are interested in height-preserving invertibility (in the first instance), we need the notion of *derivability at height n*. Axioms are derivable at height 0, and should every premiss in a rule be derivable at height *at most n*, then the conclusion of the rule will derivable at height $n + 1$.

We now come to the definitions of invertibility. Intuitively, we say that a rule is invertible if, given the conclusion of a rule, we can derive its premisses. We are most interested in the invertibility of the defining (also known as primitive) rules of the calculus. The notion we use is that of *strong admissibility* from (Dyckhoff & Negri 2000), which is also known as depth-preserving admissibility (Negri & von Plato 2001):

Definition 5 (Strong Admissibility) *The rule R given by:*

$$\frac{S}{S'}\ R$$

is ***strongly admissible*** *in a calculus iff for every n and every derivation of height n of an instance of S there is a derivation of height $\leq n$ of the corresponding instance of S'.* ⊣

There is a corresponding notion of *admissibility* which drops the requirement that the derivation of S' have height not greater than that of S. This definition shows that we prove a rule is strongly admissible by taking an arbitrary instance of the rule and showing that this instance has the necessary properties.

Definition 6 (Invertible rule) *For a calculus defined by a set of primitive rules $\mathcal{R}$, we say that a rule $R \in \mathcal{R}$ with premisses $P_1, P_2, \ldots, P_n$ and conclusion C, is* ***(strongly) invertible with respect to*** $\mathcal{R}$ *if, for each premiss P_i, the rule:*

$$\frac{C}{P_i}$$

is (strongly) admissible.

If every such $R \in \mathcal{R}$ is invertible, we say that $\mathcal{R}$ is invertible. ⊣

All of the strong admissibility results we prove require the strong admissibility of weakening (this is more commonly known as depth-preserving weakening). We give a lemma that guarantees that weakening is depth-preserving, but require the following definition to do so (Negri 2005), (Rasga 2007):

Definition 7 (Context Dependent Rules) *A* ***context dependent rule*** *is a rule which has side conditions which place restrictions upon the context of formulae allowable for instantiations of that rule.* ⊣

Many rules with variable binding are context dependent rules, as is the usual rule for necessitation on the right in modal logic (Troelstra & Schwichtenberg 2000):

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Box\Gamma, \Gamma' \Rightarrow \Box\phi, \Diamond\Delta, \Delta'}$$

where here the side condition is that all formulae in the antecedent must be boxed. Note that context dependent rules have a necessarily larger active part than a non-context dependent rule.

3 Weakening

We have the following, simple result:

Lemma 1 (dp-Weakening) *Suppose $\mathcal{R}$ is a calculus containing generalised axioms of the form:*

$$\overline{\Gamma, P \Rightarrow P, \Delta}$$

and further that $\mathcal{R}$ has no context-dependent rules, then weakening is strongly admissible in $\mathcal{R}$. That is, the rule:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$.

Proof. A routine induction. Since there are no context dependent rules, we know for instance there will be no stipulations of the form $|\Gamma| \leq n$ for some natural number n, or the requirement that all formulae be boxed (for example). So, it follows immediately that if $\Gamma \Subset \Gamma'$, where $\Subset$ is the subset relation for multisets, and a rule was applicable with Γ, then it will be applicable with Γ', because the active part of the rule is unaffected. $\dashv$

That there are *no* context dependent rules is very restrictive. In particular, it rules out lots of first order rules. This restriction can be relaxed and will be in §7.

4 Multisuccedent Calculi

We have two sets of conditions, one for right rules being invertible, and one for left rules being invertible. After the proofs, we will give examples of rules which satisfy the conditions. The form of the lemmata may seem odd; however, the additional multisets of formulae in the premisses come from the active part of the rule.

4.1 Right rules

Lemma 2 *Let $\mathcal{R}$ be a set of decomposable rules defined by a set of monoprincipal rules. Then, the rule:*

$$\frac{\Gamma \Rightarrow \star_s(\vec{\phi}), \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$ if $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every rule with $\star_s(\vec{\phi})$ principal on the right.

Proof. Let $\Gamma \Rightarrow \Delta \oplus \star_s(\vec{B})$ be an instantiation of the premiss. We prove the lemma by induction on the height n of the derivation of $\Gamma \Rightarrow \star_s(\vec{B}) \oplus \Delta$. If $n = 0$, then there exists some atom p such that $p \in \Gamma$ and $p \in \Delta \oplus \star_s(\vec{B})$, or $\bot \in \Gamma$. We then have $p \in \Gamma + \Gamma'$ and $p \in \Delta + \Delta'$, or $\bot \in \Gamma + \Gamma'$, and so $\Gamma + \Gamma' \Rightarrow \Delta + \Delta'$ is an axiom.

If $n > 0$, then we do case analysis on the last inference, r, in the derivation of $\Gamma \Rightarrow \Delta \oplus \star_s(\vec{B})$. There are four subcases to consider:

1. r was an instance of a normal rule, and $\star_s(\vec{B})$ is principal for r.
2. r was an instance of an IW rule, and $\star_s(\vec{B})$ is principal for r.
3. r was an instance of a normal rule, and $\star_s(\vec{B})$ is not principal for r.
4. r was an instance of an IW rule, and $\star_s(\vec{B})$ is not principal for r.

Case 1. From the assumptions, $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of r, and so $\Gamma + \Gamma' \Rightarrow \Delta + \Delta'$ is a premiss of r. This means it is derivable at height $n-1$, and thus at a height not greater than n, as required.
Case 2. The conclusion of r is of the form:

$$\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \star_s(\vec{B})$$

Since $\star_s(\vec{B})$ is principal for r, the sequent $\Gamma_1 + \Gamma' \Rightarrow \Delta_1 + \Delta'$ is a premiss of r, and thus is derivable at height $n-1$. We use dp-weakening to obtain:

$$\Gamma_1, \Gamma_2, \Gamma' \Rightarrow \Delta_1, \Delta_2, \Delta'$$

as derivable at height $n-1$, but this is just $\Gamma + \Gamma' \Rightarrow \Delta + \Delta'$, and so we are done.
Case 3. There are two further cases here; one where r was an instance of a left rule, and one where r was an instance of a right rule. We show the latter; the former is simpler. Suppose r had principal formula $\circ_t(\vec{D})$, then r is of the form:

$$\frac{\Gamma'', \Gamma''_1 \Rightarrow \Delta'', \Delta''_1 \quad \cdots \quad \Gamma'', \Gamma''_n \Rightarrow \Delta'', \Delta''_n}{\Gamma'' \Rightarrow \circ_t(\vec{D}), \Delta''}$$

We have that $\Gamma \equiv \Gamma''$, and $\Delta \oplus \star_s(\vec{B}) \equiv \Delta'' \oplus \circ_t(\vec{D})$. From this we have a $\Delta^\sim$ such that:

1. $\Delta = \Delta^\sim \oplus \circ_t(\vec{D})$
2. $\Delta'' = \Delta^\sim \oplus \star_s(\vec{B})$

Rewriting r with 2, we see that $\star_s(\vec{B})$ is present in *every* premiss of r, and we can therefore apply the induction hypothesis at the lower height of the premisses. Thus, we have:

$$\frac{\dfrac{\Gamma, \Gamma''_1 \Rightarrow \Delta''_1, \Delta^\sim, \star_s(\vec{B})}{\Gamma, \Gamma''_1, \Gamma' \Rightarrow \Delta''_1, \Delta^\sim, \Delta'} \quad \cdots \quad \dfrac{\Gamma, \Gamma''_n \Rightarrow \Delta''_n, \Delta^\sim, \star_s(\vec{B})}{\Gamma, \Gamma''_n, \Gamma' \Rightarrow \Delta''_n, \Delta^\sim, \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta^\sim, \circ_t(\vec{D}), \Delta'}$$

Using the equation 1 we have the result.
Case 4. r is an instance of an IW rule, say R; suppose $\circ_t(\vec{D})$ was principal for r on the right (the left case is similar). Then, r is of the form:

$$\frac{\Gamma^1, \Gamma''_1 \Rightarrow \Delta^1, \Delta''_1 \quad \cdots \quad \Gamma^1, \Gamma''_n \Rightarrow \Delta^1, \Delta''_n}{\Gamma^1, \Gamma^2 \Rightarrow \Delta^1, \Delta^2, \circ_t(\vec{D})}$$

There are two subcases; one where $\star_s(\vec{B})$ is in Γ^1 and another where it is in Γ^2. Consider the former. We can then apply the induction hypothesis to every premiss of r, after which we can another instance of R, say r', to this new set of premisses, and we complete the case in the same manner as in case 3, above.

In the latter, $\star_s(\vec{B})$ was part of the implicit weakening of r. Thus, there is some Δ^3 such that $\Delta^2 = \Delta^3 \oplus \star_s(\vec{B})$, and so the conclusion of r is rewritten as:

$$\Gamma^1, \Gamma^2 \Rightarrow \Delta^1, \Delta^3, \star_s(\vec{B}), \circ_t(\vec{D})$$

Taking the premisses of r, apply a new inference, r', based on R, to them; r' is the same as r except that the implicit weakening is $\Gamma^2 + \Gamma'$ on the left and $\Delta^3 + \Delta'$ on the right. We then have:

$$\Gamma^1, \Gamma^2, \Gamma' \Rightarrow \Delta^1, \Delta^3, \Delta', \circ_t(\vec{D})$$

is derivable at height n. We also know, from comparing the two forms for r, that:

1. $\Gamma^1, \Gamma^2 = \Gamma$
2. $\Delta^1, \Delta^2, \circ_t(\vec{D}) = \Delta, \star_s(\vec{B})$

and so, rewriting the second to include Δ^3, we have shown:

$$\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$$

is derivable at height n, which completes the case, and the proof. ⊣

4.1.1 Examples

Consider the calculus **G3cp**. All of the right rules are invertible, because the following rules are all strongly admissible:

$$\frac{\Gamma \Rightarrow \phi \wedge \psi, \Delta}{\Gamma \Rightarrow \phi, \Delta} \qquad \frac{\Gamma \Rightarrow \phi \wedge \psi, \Delta}{\Gamma \Rightarrow \psi, \Delta}$$

$$\frac{\Gamma \Rightarrow \phi \vee \psi, \Delta}{\Gamma \Rightarrow \phi, \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \phi \supset \psi, \Delta}{\Gamma, \phi \Rightarrow \psi, \Delta}$$

Take, for instance, the left premiss of $R\wedge$. We have that $\Gamma' = \emptyset$, and $\Delta' = \phi$. Furthermore, the only rule which can have a conjunction principal on the right is $R\wedge$, and so $\Gamma \Rightarrow \Delta \oplus \phi$ will be a premiss of the rule.

However, when we consider a multisuccedent version of **G3ip**, we usually have the following rule for implication on the right (see, for instance (Troelstra & Schwichtenberg 2000)):

$$\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \supset \psi, \Delta}$$

Clearly, this is only invertible if $\Delta = \emptyset$, which is not true in general, and it only satisfies the conditions of our lemma if $\Delta = \emptyset$.

Note that we talk about *invertibility* of the rules for **G3cp**. Suppose a calculus contained an IW rule, R. The lemmata of this section would *not* give the invertibility of R; each premiss of R is not shown derivable at height not greater than the conclusion of R by an application of the lemma. Rather, a weakened version of each premiss is derivable at a height not greater than the height of the conclusion of R. More concrete examples of this will be given in §8.

The condition for the applicability of the lemma is easily verifiable, for a finite set of rules. In fact, for every finite set of rules, the conditions for all of our lemmata are easily verifiable.

4.2 Left Rules

Here, we get to appeal to symmetry with the multisuccedent right rule case.

Lemma 3 *Let $\mathcal{R}$ be a set of decomposable rules defined by a set of monoprincipal rules. Then, the rule:*

$$\frac{\Gamma, \star_s(\vec{\phi}) \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$ if $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every rule with $\star_s(\vec{\phi})$ principal on the left.

Proof. The proof is symmetrical to that of §4.1. ⊣

4.2.1 Examples

The rule for implication on the left for **G3ip** is not invertible (see §5.2), however the multisuccedent version of the calculus has the following form:

$$\frac{\Gamma, \phi \supset \psi \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \supset \psi \Rightarrow \Delta}$$

This is strongly invertible; for each premiss, the lemma is applicable. In the left case, $\Gamma' = \phi \supset \psi$ and $\Delta' = \phi$, and in the right case $\Gamma' = \psi$ and $\Delta' = \emptyset$. Furthermore, any rule with $\phi \supset \psi$ principal on the left must be $L\supset$; no other rules have $\phi \supset \psi$ principal on the left.

5 Single succedent calculi

We have the restriction that the succedents contain at most one metaformula. This means that if we try to apply the previous lemmata, Δ' is empty.

5.1 Right rules

Lemma 4 *Let $\mathcal{R}$ be a set of decomposable rules defined by a set of monoprincipal rules restricted to single formula succedents. Then, the rule*

$$\frac{\Gamma \Rightarrow \star_s(\vec{\phi})}{\Gamma, \Gamma' \Rightarrow \phi}$$

is strongly admissible in $\mathcal{R}$ if $\Gamma' \Rightarrow \phi$ is a premiss of the active part of every rule with $\star_s(\vec{\phi})$ principal on the right.

Proof. An immediate application of the lemma of §4.1. ⊣

5.1.1 Examples

Take the standard formulation of **G3ip** from (Troelstra & Schwichtenberg 2000). Then, consider the rules $R\supset$ and $R\wedge$. $R\supset$:

$$\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \supset \psi}$$

is invertible; we have the conditions satisfied, because $\phi \supset \psi$ is principal on the right only for this rule, and $\Gamma' = \phi$. Thus, the rule is invertible, as expected.

Both premisses of $R\wedge$:

$$\frac{\Gamma \Rightarrow \phi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}$$

are derivable at a lower height than the conclusion, because, given a rule with $\phi \wedge \psi$ principal on the right, $\emptyset \Rightarrow \phi$ and $\emptyset \Rightarrow \psi$ will be premisses of the active part of this rule.

By contrast, consider the rules for disjunction on the right:

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}$$

Since neither $\Gamma \Rightarrow \psi$ nor $\Gamma \Rightarrow \phi$ is a premiss of *every* rule which has $\phi \vee \psi$ principal on the right, then our lemma says nothing about the invertibility of either rule.

5.2 Left Rules

Lemma 5 *Let $\mathcal{R}$ be a set of decomposable rules defined by a set of monoprincipal rules restricted to single formula succedents. Then, the rule*

$$\frac{\Gamma, \star_s(\vec{\phi}) \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \psi}$$

is strongly admissible in $\mathcal{R}$ if $\Gamma' \Rightarrow \psi$ is a premiss of the active part of every rule with $\star_s(\vec{\phi})$ principal on the left.

Proof. A simple application of Lemma 3, with the restriction to single metaformulae in succedents. ⊣

5.2.1 Examples

Most of the rules of **G3ip** are shown to be strongly invertible by this lemma. For example:

$$\frac{\Gamma, \phi \wedge \psi \Rightarrow \gamma}{\Gamma, \phi, \psi \Rightarrow \gamma}$$

can be shown to be strongly admissible; take $\Gamma' = \wr \phi, \psi \int$ and $\Delta' = \emptyset$.

The left implication rule for **G3ip**:

$$\frac{\Gamma, \phi \supset \psi \Rightarrow \phi \quad \Gamma, \psi \Rightarrow \gamma}{\Gamma, \phi \supset \psi \Rightarrow \gamma}$$

does not satisfy the conditions of the lemma. The right premiss is derivable at a lower height than the conclusion, but the left premiss is not. The left premiss fails our conditions, owing to its having a different succedent to its conclusion.

6 Extensions

6.1 Rule Sets

We stated in §1 that the lemmata in (Dawson 2008) were logical consequences of our lemmata. Dawson is concerned with the admissibility of certain rules, rather than their strong admissibility. As is suggested by the names, strong admissibility implies admissibility. He is also investigating the invertibility of a set of rules, rather than the derivability of a given premiss of a rule. We require the following definition, of what it means for a set of rules to have the *unique conclusion property*:

Definition 8 (Unique Conclusion Property) *A set of decomposable rules $\mathcal{R}$ has the* ***unique conclusion property*** *iff for all S and T in $\mathcal{R}$, if the conclusion of the active part of S is the same as the conclusion of the active part of T, then $S = T$.* ⊣

The lemma he proves is as follows:

Lemma 6 *Let $\mathcal{R}$ be a set of decomposable rules defining a sequent calculus. If $\mathcal{R}$ has the unique conclusion property, $(\vec{P}, C)$ is an instance of a rule of $\mathcal{R}$ and C is derivable with respect to $\mathcal{R}$, then so is every $p \in \vec{P}$.*

Proof. Suppose that every rule has a unique conclusion, and further that $(\vec{P}, C)$ is an inference in the calculus and C is derivable with respect to the calculus. Let p be a premiss of this inference (i.e. $p \in \vec{P}$). From the fact that every rule has a unique conclusion we have that p is a premiss of *every* inference with conclusion C. Then depending on the particular calculus formation or form of C, we apply one of the lemmata from the previous sections, and so p is derivable. ⊣

What we actually prove is the stronger notion that the premiss is strongly admissible, not just admissible.

6.2 More complex propositional calculi

As noted in §2, the rule $L0\supset$ from **G4ip** precludes that calculus from the analysis in the previous sections. Here, we relax the condition that the active part of rules may only have one compound formula in their conclusions. Now, however, *every* metaformula which appears in the conclusion of the active part is principal for the rule. As an example, both P and $P \supset \phi$ are principal for:

$$\frac{\Gamma, P, \phi \Rightarrow \gamma}{\Gamma, P, P \supset \phi \Rightarrow \gamma}$$

We can then reconstruct similar lemmata to those of the previous sections, however we must be careful with respect to atoms and $\bot$. We have the following:

Lemma 7 *Let $\mathcal{R}$ be a set of decomposable rules. Then, the rule:*

$$\frac{\Gamma, \Gamma'' \Rightarrow \Delta, \Delta''}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible if:

1. *For every $A \in \Gamma'', \Delta''$, we have $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every rule with A principal.*
2. *Every atom variable in Γ'' occurs in Γ'.*
3. *Every atom variable in Δ'' occurs in Δ'.*
4. *If $\bot \in \Gamma''$, then $\bot \in \Gamma'$.*

Proof. Let $\Gamma + \Gamma'' \Rightarrow \Delta + \Delta''$ by an instantiation of the premiss. We proceed by induction on the height, n, of the derivation of the premiss. If $n = 0$, then the premiss was an axiom, so there is some atom p such that $p \in \Gamma + \Gamma''$ and $p \in \Delta + \Delta''$, or $\bot \in \Gamma + \Gamma''$. Conditions 2,3 and 4 guarantee that $p \in \Gamma + \Gamma'$ and $p \in \Delta + \Delta'$ in the first case, and $\bot \in \Gamma + \Gamma'$ in the second; the conclusion is likewise an axiom.

If $n > 0$, let r be the last inference used in the derivation. Then there are four cases:

1. r is an instance of a normal rule, and there is some $A \in \Gamma'', \Delta''$ which is principal for r.
2. r is an instance of an IW rule, and there is some $A \in \Gamma'', \Delta''$ which is principal for r.
3. r is an instance of a normal rule, and there is no $A \in \Gamma'', \Delta''$ which is principal for r.
4. r is an instance of an IW rule, and there is no $A \in \Gamma'', \Delta''$ which is principal for r.

Case 1. We have that $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of r. Therefore, since r is an instance of a normal rule, $\Gamma + \Gamma' \Rightarrow \Delta + \Delta'$ is derivable at height $n - 1$ and so is derivable at height at most n, as required.
Case 2. This case is similar to the equivalent case in the lemmata in sections §4.1 and §4.2.
Case 3. Since *no* formula from Γ'', Δ'' is principal for r, all such formulae must be in the context of r. Then, we reason in the same fashion as in §4.1 and §4.2.
Case 4. We know that *no* formulae from Γ'', Δ'' are principal for r. However, given r will be of the form:

$$\frac{\Gamma^1, \Gamma''_1 \Rightarrow \Delta^1, \Delta''_1 \quad \cdots \quad \Gamma^1, \Gamma''_n \Rightarrow \Delta^1, \Delta''_n}{\Gamma^1, \Gamma^2, \Gamma''_\star \Rightarrow \Delta^1, \Delta^2, \Delta''_\star}$$

we can have Γ'' is in Γ^1, or Γ^2, or partially contained in both. We reason by cases. Either, $\Gamma'' \Subset \Gamma^2$, and $\Delta'' \Subset \Delta^2$, or at least some part of Γ'' or Δ'' is in Γ^1 or Δ^1, respectively.

In the former case, the situation is similar to that in §4.1 and §4.2; we use a new inference which is the same as r except that the implicit weakening part of the inference contains Γ' and Δ', instead of Γ'' and Δ''.

In the other case, we perform two steps. Firstly, we weaken every premiss of r so that the context of each premiss contains Γ'' and Δ''. This will be the same weakening for every premiss, because every premiss contains Γ^1 and Δ^1, and hence the same elements of Γ'' and Δ''. Then, we apply the induction hypothesis to each of these new premisses, so that we remove Γ'' and Δ'' and replace them with Γ' and Δ' in the context of each premiss.

Then, we remove from Γ^2 and Δ^2 any formulae from Γ'' and Δ'', respectively, and we will use this as our new implicit weakening. Applying the new inference with this information will yield:

$$\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$$

as derivable at height n, and this completes the case, and the proof of the lemma. ⊣

In particular, we have the results from the previous sections are now specialisations of this result.

6.2.1 Examples

When restricted to single formulae in the obvious way (as in §5) on the right, **G4ip** is a calculus in which all rules have the form given above. The rule which did not allow **G4ip** to be classified as a monoprincipal calculus was $L0\supset$. Now we can see, however, that it is indeed invertible:

$$\frac{\Gamma, P, \phi \Rightarrow \gamma}{\Gamma, P, P \supset \phi \Rightarrow \gamma}$$

The atom variable P is retained in the premiss, and this rule is the only rule where $P \supset \phi$ will be principal on the left.

6.3 Rules which fail the conditions

Some rules fail our conditions, but nevertheless are invertible. Suppose we augment **G3cp** with two extra rules for conjunction on the left, giving us the following three rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ L\wedge_1 \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ L\wedge_2$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ L\wedge$$

Now, the rule

$$\frac{\Gamma, \phi \wedge \psi \Rightarrow \Delta}{\Gamma, \phi, \psi \Rightarrow \Delta}$$

is strongly admissible in this calculus, however this is not from an application of the lemma in §4.2. We cannot apply that lemma since $\wr \phi, \psi \wr \Rightarrow \emptyset$ is *not* a premiss of the active part of every rule with $\phi \wedge \psi$ principal on the left. We can show invertibility directly, though, since if $A \wedge B$ was principal on the left for the last inference, then the last inference was an instance of $L\wedge$, in which case we are done, or it was an instance of $L\wedge_1$ or $L\wedge_2$. In either case, we simply weaken with the appropriate formula (B and A respectively), and we have completed the proof of the principal case. The non-principal and axiom cases are as before, for instance in §4.1.

It is a simple matter to extend the conditions so that we can capture the invertibility of rules such as $L\wedge$ above. We change the condition "$\Gamma' \Rightarrow \Delta'$ *is a premiss of the active part of every rule...*" to

> $\Gamma' \Rightarrow \Delta'$ can be obtained, by weakening, from a premiss of the active part of every rule...

7 First-Order Calculi

The conditions under which we performed analysis in the previous sections were quite restrictive. In particular, we ruled out, in §2, first-order calculi and modal logics under the proviso of "no context dependent rules". We seek now to relax this condition, in the first instance to permit our lemmata to be applicable to first-order calculi. We need to change our definition of formulae from one involving purely propositional connectives.

We need to show weakening is strongly admissible in the presence of free and bound variables. The intuitive solution is to rename all of the variables in a derivation so that there are no clashes with any variables in the formulae with which we weaken. This requires a substitution lemma to be strongly admissible. We need some additional definitions. The first is found in (Rasga 2007):

Definition 9 (Freshness Proviso) *Suppose R is a rule with antecedent Γ, succedent Δ and some distinguished formula A. Such a rule is said to have a **freshness proviso** iff it has a side condition of the form:*

$$y \textit{ fresh for } \Gamma, y \textit{ fresh for } \Delta, y \textit{ fresh for } A$$

⊣

In addition to the rule sets from the propositional cases, i.e. axioms and monoprincipal rules, we also have two new rule sets. The first contains those rules which have a single, first-order formula in the conclusion of the active part. The second contains those rules which have a single, first-order formula in the conclusion of the active part *and* a freshness proviso on the variable which is bound.

We use a notion from (Zamansky & Avron 2006) to give a general form for quantifiers. This was also used in (Ciabattoni & Terui 2006*a*):

Definition 10 (General Quantifiers) *An (n, k)-ary quantifier for $n > 0$, $k \geq 0$ is a generalised logical connective, which binds k variables and connects n formulae.* ⊣

As an example, $\wedge$ can be seen as a $(2, 0)$-ary quantifier, as indeed can $\vee$ and $\supset$. The usual first order $\forall$ and $\exists$ are $(1, 1)$-ary quantifiers. The bounded universal ($\bar{\forall}$) and existential ($\bar{\exists}$) quantifiers given by:

$$\bar{\forall}x.(p(x), q(x)) \equiv \forall x.(p(x) \supset q(x))$$

$$\bar{\exists}x.(p(x), q(x)) \equiv \exists x.(p(x) \wedge q(x))$$

are $(2, 1)$-ary quantifiers. The rules for these quantifiers in an extension of **G3c** are a combination of the rules for $\forall$ and $\supset$, for example:

$$\frac{\Gamma, [y/x]\phi \Rightarrow [y/x]\psi, \Delta}{\Gamma \Rightarrow \bar{\forall}x.(\phi, \psi), \Delta}$$

where y is fresh for Γ, Δ. We use similar notation to that described in §2 for such quantifiers: $\nabla_{m,k}\vec{x}.(\vec{\phi})$ is an (m, k)-ary quantifier where $\vec{x}$ is a k-tuple of variables, and $\vec{\phi}$ is an m-tuple of metaformulae.

Lemma 8 (Substitution lemma) *Given a calculus defined by a set of primitive rules $\mathcal{R}$ which can contain freshness provisos, if y is a variable which is fresh for $\Gamma \Rightarrow \Delta$, then the rule:*

$$\frac{\Gamma \Rightarrow \Delta}{[y/x]\Gamma \Rightarrow [y/x]\Delta}$$

is strongly admissible in the calculus.

Proof. A standard induction on the height of the derivation of an instance of $\Gamma \Rightarrow \Delta$. ⊣

We then have our required extension to the weakening result from §3:

Lemma 9 (Weakening - First Order) *If a calculus contains a general axiom $\Gamma \Rightarrow \Delta$ where there is some atom variable P such that $P \in \Gamma$ and $P \in \Delta$, and there are no context dependent rules in the calculus other than freshness provisos, then weakening is strongly admissible.*

Proof. A standard induction as in §3. We appeal to the Substitution Lemma when the last inference used in a derivation was an instance of a rule with a freshness proviso, to ensure that the new formula introduced by the weakening is suitably fresh for the derivation. ⊣

Intuitively, a premiss to be derivable given the derivation of the conclusion of a rule if the freshness provisos are observed, and there are no new *specific* substitutions in the premiss. We want there to be only new *general* substitutions in the premiss, where the substituted variable can range over the entire domain for which it is defined. In other words, new substitutions such as $[y/x]$ are acceptable in a premiss, whereas $[t/x]$ is not, where t is some term.

We are now ready to extend the results from §4.1 and §4.2. We need only consider the cases where there is some variable binding, in other words the cases of (n, k)-ary connectives for which $k > 0$; the rest of the cases will be as before.

Lemma 10 (First-Order Right Rules) *The rule:*

$$\frac{\Gamma \Rightarrow \nabla_{m,k}\vec{x}.(\vec{\phi}), \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible if:

1. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every rule with $\nabla_{m,k}\vec{x}.(\vec{\phi})$ principal on the right.*
2. *Γ' and Δ' contain no specific substitutions.*
3. *All freshness provisos present in the (primitive) rules are observed.*

Proof. By induction on the height, n, of a derivation ending with an instance of $\Gamma \Rightarrow \nabla_{m,k}\vec{x}.(\vec{B}) \oplus \Delta$. The proof is much the same as that in §4.1, except that condition 3 ensures that if the last inference used was non-principal for a (m', k')-ary quantifier, where $k' > 0$, then we can apply this inference after applying the induction hypothesis. Uses of the Substitution Lemma may be needed to rename variables to guarantee condition 3 holds. Condition 2 means that there are no new arbitrary terms in Γ' and Δ'. ⊣

Lemma 11 (First-Order Left Rules) *The rule:*

$$\frac{\Gamma, \nabla_{m,k}\vec{x}.(\vec{\phi}) \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible if:

1. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every rule with $\nabla_{m,k}\vec{x}.(\vec{\phi})$ principal on the left.*
2. *Γ' and Δ' contain no specific substitutions.*
3. *All freshness provisos present in the (primitive) rules are observed.*

Proof. The proof is symmetrical to that of the previous lemma. ⊣

7.1 Examples

We consider the four rules for $(1, 1)$-ary quantifiers from **G3c**:

$$\frac{\Gamma, [t/x]\phi, \forall x.\phi \Rightarrow \Delta}{\Gamma, \forall x.\phi \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow [y/x]\phi, \Delta}{\Gamma \Rightarrow \forall x.\phi, \Delta}\ R\forall$$

$$\frac{\Gamma, [y/x]\phi \Rightarrow \Delta}{\Gamma, \exists x.\phi \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow [t/x]\phi, \exists x.\phi, \Delta}{\Gamma \Rightarrow \exists x.\phi, \Delta}\ R\exists$$

where y is fresh for the conclusions of $R\forall$ and $L\exists$. As is easily verified, $R\forall$ and $L\exists$ satisfy the conditions of the lemma and so are strongly invertible. As an example, the rule:

$$\frac{\Gamma \Rightarrow \forall x.\phi, \Delta}{\Gamma \Rightarrow [y/x]\phi, \Delta}$$

is shown to be strongly admissible. Let $\Delta' = \forall x.\phi$, and $\Gamma' = \emptyset$. There are no specific substitutions in Δ' and, since $R\forall$ is the only rule in which an universal formula is principal on the right, we have strong admissibility, and hence the strong invertibility of $R\forall$.

In both $L\forall$ and $R\exists$, Γ' or Δ' contains specific substitutions. It could be argued that both premisses are derivable from the conclusion *at the same height* by an application of depth-preserving weakening, however, this suggests we know which term t to use in the weakened formula, which in general is not possible. One could, therefore, argue that our conditions are certainly not necessary, since we have an example where they are not satisfied, but the rule is strongly invertible. We cannot even appeal to weakening for $R\exists$, however, when we consider the single succedent version of this rule[1]:

$$\frac{\Gamma \Rightarrow [t/x]\phi}{\Gamma \Rightarrow \exists x.\phi}\ R\exists$$

As a further example, consider the four rules for the bounded quantifiers $\bar{\forall}$ and $\bar{\exists}$, whose definitions were given above and in (Zamansky & Avron 2006):

$$\frac{\Gamma, [t/x]\phi \Rightarrow \Delta \quad \Gamma, [t/x]\psi \Rightarrow \Delta}{\Gamma, \bar{\forall}x.(\phi, \psi) \Rightarrow \Delta}\ L\bar{\forall}$$

$$\frac{\Gamma, [y/x]\phi \Rightarrow [y/x]\psi, \Delta}{\Gamma \Rightarrow \bar{\forall}x.(\phi, \psi), \Delta}\ R\bar{\forall}$$

$$\frac{\Gamma, [y/x]\phi, [y/x]\psi \Rightarrow \Delta}{\Gamma, \bar{\exists}x.(\phi, \psi) \Rightarrow \Delta}\ L\bar{\exists}$$

$$\frac{\Gamma \Rightarrow [t/x]\phi, \Delta \quad \Gamma \Rightarrow [t/x]\psi, \Delta}{\Gamma \Rightarrow \bar{\exists}x.(\phi, \psi)\Delta}\ R\bar{\exists}$$

[1] Note we must make appropriate restrictions upon Δ and Δ' to ensure the calculus is single succedent.

where y is fresh for the conclusions of $R\bar{\forall}$ and $L\bar{\exists}$. These two rules are also strongly invertible, as can be easily checked. However, the premisses of $L\bar{\forall}$ and $R\bar{\exists}$ contain specific substitutions, and so we cannot conclude anything about their invertibility using the methods outlined here.

8 Modal Logic

In the previous section, we allowed a certain kind of context-dependent rule, namely one with a freshness proviso. In this section, we allow a different kind of context-dependent rule, one with *modalised contexts*. We call a multiset of formulae *modalised* if it is of the form $!\Gamma$, where $!$ is some modal operator.

Definition 11 (Modalised Context Rules) *A rule is a modalised context rule iff:*

1. *Any instantiation of that rule has modalised multisets in its active part.*
2. *For any multiset Γ, an appropriately modalised Γ appears in the active part of an instantiation of the rule.*

$\dashv$

We have that $L\Diamond$, for a classical calculus extended with modalities (Troelstra & Schwichtenberg 2000), is a modalised context rule:

$$\frac{\Box\Gamma, \phi \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond\phi, \Gamma' \Rightarrow \Diamond\Delta, \Delta'}$$

whereas $R\Diamond$ is not:

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \Diamond\phi}$$

If we take $\Gamma \neq \emptyset$, and[2] $\Gamma \neq \phi$, then for any modal operator $!$, we do not have that $!\Gamma$ appears in the active part of the rule. We call rules which are not modalised context rules, but which nevertheless have a modal metaformula in the active part of the rule, *basic modal rules.*

Modalised context rules have large active parts. From clause 2 in the definition above, these active parts could be further decomposed into a context consisting of the arbitrarily instantiated multisets (appropriately modalised) and what we call the *prime part* of the rule. Formally, we have the following:

Definition 12 (Prime Formulae, Prime Part) *A metaformula ϕ is* ***prime*** *for the active part of a modalised context rule iff:*

1. *ϕ cannot be instantiated with an appropriately modalised arbitrary multiset, OR*
2. *ϕ is the submetaformula occurrence of a prime metaformula.*

The ***prime part*** *of a modalised context rule is the active part of the rule where all non-prime metaformulae have been deleted.* $\dashv$

Similarly to §2, a *prime formula of an inference* is an instantiation of the prime metaformula of the corresponding rule.

[2] Treating ϕ as a singleton multiset.

In $L\Diamond$ above, the prime part of the rule is:

$$\frac{\phi \Rightarrow}{\Diamond\phi \Rightarrow}$$

This definition is similar to the *quasi-active formulae* in (Lutovac & Harland 2005). The modalised contexts are necessary, but not sufficient, for the application of a modalised context inference.

We add to our definition of (meta)formulae some modal operators. These will be represented in the analysis by $!(\vec{B})$ and $\bullet(\vec{D})$, where the lengths of $\vec{B}$ and $\vec{D}$ correspond to the arities of $!$ and $\bullet$, respectively.

As has been done in previous sections, we prove the strong admissibility of weakening for such sequent calculi.

Lemma 12 (Modal dp-Weakening) *Let $\mathcal{R}$ be a calculus containing monoprincipal propositional rules, basic modal rules, and modalised context rules. If every modalised context rule in $\mathcal{R}$ is an IW rule, then the rule:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$.

Proof. Let $\Gamma \Rightarrow \Delta$ be an instantiation of the premiss. The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ and is standard. In the case where the last inference is a modalised context inference, then we do not use the induction hypothesis, we simply apply a new inference with a suitable extension of the conclusion. $\dashv$

Using the definition of principal from §6, where every metaformula in the active part of a conclusion is principal, we can keep the conditions very close to those of §4.1 and §4.2. Note two inferences can differ in the formulae occurring in the modalised multisets. Were we to use the same conditions as in §4.1, we would encounter a problem; for instance $\Diamond A$ is principal on the left for many instances of the rule, $L\Diamond$, given above.

Lemma 13 (Right modal rules) *Let $\mathcal{R}$ be a calculus containing monoprincipal propositional rules, basic modal rules, and modalised context rules. The rule*

$$\frac{\Gamma \Rightarrow !(\vec{\phi}), \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$ if

1. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the prime part of every modalised context rule in which $!(\vec{\phi})$ is principal on the right.*
2. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every basic modal rule in which $!(\vec{\phi})$ is principal on the right.*
3. *All modalised context rules in $\mathcal{R}$ are IW rules.*

Proof. Let $\Gamma \Rightarrow \Delta \oplus !(\vec{B})$ be an instance of the premiss of the above rule. We proceed by induction on the height n of the derivation of this premiss. If $n = 0$, or $n > 0$ and the last inference was a propositional rule, then we proceed as in §4.1 ($!(\vec{B})$ will never be principal for such an inference). If the last inference r was an instance of a modal rule, say R, there are four cases:

1. $!(\vec{B})$ is principal for r, and r is a basic modal inference.
2. $!(\vec{B})$ is principal for r, and r is a modalised context inference.
3. $!(\vec{B})$ is non-principal for r, and r is a basic modal inference.
4. $!(\vec{B})$ is non-principal for r, and r is a modalised context inference.

Case 1. The result is immediate, from condition 2.
Case 2. $\Gamma \Rightarrow \Delta \oplus !(\vec{B})$ is the conclusion of a modalised context inference. Let $\bullet_1, \ldots, \bullet_n, !_1, \ldots, !_m$ be modal operators, then for some $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_m$ and Γ'', Δ'', we can rewrite the conclusion of r as:

$$\bullet_1\Gamma_1, \cdots, \bullet_n\Gamma_n, \Gamma'' \Rightarrow !(\vec{B}), !_1\Delta_1, \cdots, !_m\Delta_m, \Delta''$$

From condition 1, $\Gamma' \Rightarrow \Delta'$ is premiss of the prime part of r, and so:

$$\bullet_1\Gamma_1, \cdots, \bullet_n\Gamma_n, \Gamma' \Rightarrow !_1\Delta_1, \cdots, !_m\Delta_m, \Delta'$$

is a premiss of r. From condition 3, weakening is strongly admissible and thus we weaken with Γ'' and Δ'' to obtain the desired result.
Case 3. There are two further subcases, one where r is an instance of a normal rule, and one where r is an instance of an IW rule. In the former, from $!(\vec{B})$ being non-principal for r, $!(\vec{B})$ appears in every premiss of the inference, as part of the context. We can thus apply the induction hypothesis to each premiss. To this set of premisses extended with the new context involving Γ' and Δ', we apply the instance of r' of R which uses that context, and we are done. (The details are the same as in §4.1).

In the latter case, again the details are similar to the equivalent case in §4.1 and so are omitted.
Case 4. From condition 3, every modalised context rule is an IW rule, and therefore suppose that the conclusion of r was

$$\bullet_1\Gamma_1, \cdots, \bullet_n\Gamma_n, \Gamma'' \Rightarrow !_\star(\vec{D}), !_1\Delta_1, \cdots, !_m\Delta_m, \Delta''$$

for some modal operators $\bullet_1, \ldots, \bullet_n, !_1, \ldots, !_m$ and multisets $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_m$ and Γ'', Δ''. Because $!(\vec{B})$ is non-principal on the right, we must have that $!(\vec{B}) \in \Delta''$. Therefore, let $\Delta^\sim$ be such that $\Delta'' = \Delta^\sim \oplus !(\vec{B})$. We use a new instance of R which has $\Gamma'' + \Gamma'$ and $\Delta^\sim + \Delta'$ as the context, and we are done. ⊣

Lemma 14 (Left modal rules) *Let $\mathcal{R}$ be a calculus containing monoprincipal propositional rules, basic modal rules, and modalised context rules. The rule*

$$\frac{\Gamma \Rightarrow !(\vec{\phi}), \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$ if

1. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the prime part of every modalised context rule with $!(\vec{\phi})$ principal on the left.*
2. *$\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of every basic modal rule with $!(\vec{\phi})$ principal on the left.*
3. *Every modalised context rule in $\mathcal{R}$ is an IW rule.*

Proof. Symmetric to the proof for right modal rules. ⊣

It should be noted that these are not true invertibility results; we do not reconstruct the premisses of a modal rule exactly, but we derive weakened versions of the premisses of a modal rule.

8.1 Examples

Consider the calculus for classical propositional logic extended with modalities; in other words, **G3cp** together with the four rules:

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Box\Gamma, \Gamma' \Rightarrow \Box\phi, \Diamond\Delta, \Delta'}\ R\Box$$

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\ L\Box$$

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \Diamond\phi, \Delta}\ R\Diamond$$

$$\frac{\Box\Gamma, \phi \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond\phi, \Gamma' \Rightarrow \Diamond\Delta, \Delta'}\ L\Diamond$$

If a derivation in this system had root $\Gamma \Rightarrow \Delta \oplus \Box A$ at height n, then we can use the first lemma to assert that $\Gamma \Rightarrow \Delta \oplus A$ is derivable at height $\leq n$. Only inferences based on $R\Box$ have $\Box A$ principal on the right and:

$$\Rightarrow A$$

is a premiss of the prime part of every such inference; we can apply the lemma, which justifies the assertion.

However, the rule:

$$\frac{\Gamma \Rightarrow \Diamond\phi, \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

is not strongly admissible in this system, and we cannot apply our lemmata. In general:

$$\Rightarrow \phi$$

will not be a premiss of the prime part of every modalised context inference which has $\Diamond\phi$ principal on the right:

$$\frac{\Box\Gamma \Rightarrow \psi, \Diamond(\Delta, \phi)}{\Box\Gamma \Rightarrow \Box\psi, \Diamond(\Delta, \phi)}$$

has $\Diamond\phi$ principal on the right, but only $\Rightarrow \psi$ as a premiss of the prime part of the rule.

9 An Isabelle Implementation

The results we have implemented are of a slightly different nature from those given in the previous sections. There, we took a sequent calculus, extracted the active parts, and reasoned about them. In the implementation, we rather define a sequent calculus by giving the active parts and defining a sequent calculus as the extensions, with context, of the active parts. The notation is as follows; if $\mathcal{R}$ is a set of active parts, then $\mathcal{R}^\star$ is the set of extensions of the active parts. In other words, $\mathcal{R}^\star$ is the sequent calculus we have defined.

In this setting, we have approximately formalised the results from §4.1,4.2,7 *except* that we have imposed the further restriction that all rules be normal rules, and so the premisses and conclusions are extended with the same multisets of formulae. We have also formalised the result from §8. We show an example of the invertibility of **G3cp**. We have some abbreviations, so Conj $[A, B]$ is written in *Isabelle* as $A \wedge B$.

```
inductive-set G3cp
where
  conL:
  ([⦃A⦄ + ⦃B⦄ ⇒* ∅], ⦃A ∧ B⦄ ⇒* ∅) ∈ G3cp
| conR:
  ([∅ ⇒* ⦃A⦄, ∅ ⇒* ⦃B⦄], ∅ ⇒* ⦃A ∧ B⦄) ∈ G3cp
| disL:
  ([⦃A⦄ ⇒* ∅, ⦃B⦄ ⇒* ∅], ⦃A ∨ B⦄ ⇒* ∅) ∈ G3cp
| disR:
  ([∅ ⇒* ⦃A⦄ + ⦃B⦄], ∅ ⇒* ⦃A ∨ B⦄) ∈ G3cp
| impL:
  ([∅ ⇒* ⦃A⦄, ⦃B⦄ ⇒* ∅], ⦃A ⊃ B⦄ ⇒* ∅) ∈ G3cp
| impR:
  ([⦃A⦄ ⇒* ⦃B⦄], ∅ ⇒* ⦃A ⊃ B⦄) ∈ G3cp
```

We first show that this set of active parts is a subset of the monoprincipal active parts. This is straightforward:

```
lemma G3cpSubset:
shows G3cp ⊆ scRules
proof−
{
 fix ps c
 assume (ps,c) ∈ G3cp
 then have (ps,c) ∈ scRules by (induct) auto
}
thus G3cp ⊆ scRules by auto
qed
```

We show that the rule $R\supset$ is invertible in **G3cp**. We have to show that, for every rule in **G3cp**, if $A \supset B$ is principal on the right for r, then $A \Rightarrow B$ is in the premisses of r. Since there is only one rule with $A \supset B$ on the right, this is simple.

```
lemma impRInvert:
assumes (Γ ⇒* Δ ⊕ (A⊃B),n)
          ∈ derivable (G3cp ∪ idRules)*
shows ∃ m≤n. (Γ ⊕ A ⇒* Δ ⊕ B,m)
          ∈ derivable (G3cp ∪ idRules)*
proof−
have ∀ r ∈ G3cp. rightPrincipal r (A⊃B) ⟶
     (⦃A⦄ ⇒* ⦃B⦄) ∈ set (fst r)
  proof−
  {fix r
   assume a:r ∈ G3cp
   assume rightPrincipal r (A⊃B)
   then obtain Ps where b:r = (Ps,∅ ⇒* ⦃A⊃B⦄)
      by (cases r) auto
   with a have Ps = [⦃A⦄ ⇒* ⦃B⦄]
      apply (cases r) by (rule G3cp.cases) auto
   with b have (⦃A⦄ ⇒* ⦃B⦄) ∈ set (fst r)
      by auto
  }
  thus ?thesis by (auto)
  qed
with assms show ?thesis
    using rightInvertible by (auto simp add: G3cpSubset)
qed
```

It is possible to suppress output from an *Isabelle* build, see (Nipkow et al. 2005) for further details. The only details that have been suppressed are the names of additional facts passed to the *Isabelle* system, and the instantiations of variables in the applied lemma *rightInvertible*. In the LaTeX markup, we have also altered the appearance of some symbols to aid readability.

Even though the reader may be unfamiliar with the syntax of *Isabelle/Isar*, one can see that the above proof is quite succinct. As a comparison, to prove the same lemma by a direct method in *Isabelle* would take roughly 100 lines (Chapman 2008*b*), whereas here it takes 18 lines. The more rules which are present in a system, the greater the disparity between the two approaches. This is because we need to manually perform the inversion when the formula, in this case $A \supset B$, is not principal for the last inference. The more primitive rules there are in a calculus, the greater the number of non-principal cases one has to consider.

10 Conclusions and Further Work

The syntactic criteria we give for a rule to be invertible are simple. Moreover, they are general enough to be applicable in a large number of cases. We stated in §1 that we would like to be able to prove effectively Cut admissibility, and other such results, in *Isabelle*. The generic lemmata presented here eliminate the need to prove invertibility results for specific calculi by direct methods. Such results usually account for a large portion of the formalisations, for instance in (Chapman 2008*b*) the proofs of invertibility are 323 lines of proof, in a 720 line file, and in (Chapman 2008*a*) (a formalisation of (Dyckhoff & Negri 2000)), they consist of 955 lines of proof in a 2100 line theory file, where the goal is to prove the admissibility of Contraction for **G4ip**. In both cases, the seemingly trivial lemmata account for roughly half of the total size of the file.

Another obvious avenue of investigation is that of *necessary* conditions for invertibility. Such conditions are more difficult to come by: showing a sequent is not derivable at a given height is often achieved by a brute force search to find no valid derivations. However, this relies upon derivability being decidable.

There are still portions of this paper which have not been formalised. The results for §8, along with the more general result from §6.2, have not been formalised. As stated in §2, the notation for specifying formulae is clumsy for giving the rules of a sequent calculus. Were we able to give conditions for when a general axiom to be admissible, i.e.

$$\overline{\Gamma, \phi \Rightarrow \phi, \Delta}$$

for *any* ϕ, then we could introduce polymorphism into the work. Rather than have such things as axioms, however, we would rather want them as derived rules, with premisses. If they were axioms (i.e. zero premiss rules with height 0), then the base case in the proofs would fail. For, consider the case where $\Gamma \Rightarrow \Delta \oplus \star_s(\vec{A})$ was derivable at height 0, and moreover came from an axiom with $\star_s(\vec{A})$ as the distinguished formula. In general, $\Gamma \Rightarrow \Delta$ will not be derivable, and so neither will $\Gamma + \Gamma' \Rightarrow \Delta + \Delta'$.

As noted in §9, there is a mismatch between the definition of a sequent calculus in *Isabelle*, and that which we give in §2. Suppose that the active parts of a decomposable sequent calculus, $\mathcal{R}$, were denoted by $\hat{\mathcal{R}}$. Then, we would have that the two approaches outlined could be shown to be equivalent if

$$(\hat{\mathcal{R}})^\star = \mathcal{R}$$

Since a sequent calculus is given, in this sense, by a set of rules, we need to show that

1. $(\hat{\mathcal{R}})^\star \subseteq \mathcal{R}$
2. $\mathcal{R} \subseteq (\hat{\mathcal{R}})^\star$

The latter is simple to show, however the former is not immediately obvious. If we restrict $\mathcal{R}$ to consisting entirely of normal rules, then the equivalence holds.

A wider variety of calculi could be considered. Various linear logics do not fit into the framework presented in this paper. We hope to work on directions such as linear logic in the future.

We submit that, with some work in the areas suggested, *Isabelle* can be made a more effective tool for formalising results in structural proof theory. Furthermore, we submit that the results in this paper are worthwhile in their own right.

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