Standard Basis

(Notes based on Real-Time Rendering by Akenine-Moller, Haines and Hoffman)

A vector $\mathbf{v}$ can be represented as a linear combination of other vectors, or a *basis*.

The *standard basis* consists of unit vectors in the x, y and z directions

$$\mathbf{e}_x = (1,0,0), \mathbf{e}_y = (0,1,0), \mathbf{e}_z = (0,0,1)$$

Using matrix notation the standard basis is

$$\mathbf{E} = \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A vector $\mathbf{v}$ can then be expressed as

$$\mathbf{E}\mathbf{v} = \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix} \mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Arbitrary Basis

Vectors can be expressed in terms of other bases, where different vectors are used for the basis vectors.

A set of vectors $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$ is *linearly independent* if the only scalars which satisy

$$v_0\mathbf{u}_0 + ... + v_{n-1}\mathbf{u}_{n-1} = 0$$

are $v_0 = v_1 = ... = v_{n-1} = 0$.

This means no vector can be a constant times another — ruling out, for example, $\mathbf{u}_0 = (4, 3)$ and $\mathbf{u}_1 = (8, 6)$ where $\mathbf{u}_1 = 2 \times \mathbf{u}_0$.

If any vector $\mathbf{v} \in \mathbb{R}_n$ can be written in terms of a set of n linearly independent vectors $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$

$$\mathbf{v} = \sum_{i=0}^{n-1} v_i\mathbf{u}_i$$

then the vectors $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$ are said to *span* the Euclidean space $\mathbb{R}_n$.

If $v_0 = v_1 = ... = v_{n-1}$ are uniquely determined for any vector $\mathbf{v} \in \mathbb{R}_n$ then the vectors $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$ are a *basis* of $\mathbf{R}_n$.

In short, a basis is a set of n *linearly independent* vectors which spans the Euclidean space $\mathbb{R}_n$.

If the vectors of a basis $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$ are orthogonal to each other then the basis is *orthogonal*. If the vectors are also of unit length then the basis is *orthonormal*.

For an orthonormal basis $\mathbf{u}_0, ..., \mathbf{u}_{n-1}$ and a vector $\mathbf{p} = (p_0, ..., p_{n-1})$

$$p_i = \mathbf{p}.\mathbf{u}_i$$

Change of Basis

In computer graphics we sometimes need to change the *basis* of a vector.

For example, this arises in collision dynamics between particles where there is a need to change basis to work out the velocities after the collision.

Another example arises in bump mapping where *tangent space* is used.

A vector $\mathbf{w}$ can be expressed in the arbitrary basis $\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z$ as

$$\mathbf{F}\mathbf{w} = \begin{pmatrix} \mathbf{f}_x & \mathbf{f}_y & \mathbf{f}_z \end{pmatrix} \mathbf{w}$$

If $\mathbf{v} = \mathbf{w}$, that is, we are considering the same vector but represented in different bases, then

$$\mathbf{F}\mathbf{w} = \mathbf{v} \Longleftrightarrow \mathbf{w} = \mathbf{F}^{-1}\mathbf{v}$$

Thus to obtain $\mathbf{w}$ expressed in the new basis we multiply by the inverse of the basis

$$\mathbf{w} = \mathbf{F}^{-1}\mathbf{v}$$

If the basis $\mathbf{F}$ is orthonormal then a result from matrix theory says

$$\mathbf{F}^{-1} = \mathbf{F}^T$$

Then we have

$$\mathbf{w} = \mathbf{F}^T\mathbf{v} = \left(\begin{array}{c} \mathbf{f}_x^T \\ \mathbf{f}_y^T \\ \mathbf{f}_z^T \end{array} \right) \mathbf{v}$$

Tangent Space

Bump mapping calculations can sometimes be simplified if lighting is performed in *tangent space*.

Tangent space uses a $\mathbf{TBN}$ basis defined by a tangent $\mathbf{T}$, a *binormal* $\mathbf{B}$ and the normal $\mathbf{N}$ to a point on a surface.

The partial derivatives in the direction of the surface parameters u and v (or s and t) are two vectors $\mathbf{P_u}$ and $\mathbf{P_v}$. They both lie in the tangent plane, although are not necessarily orthogonal to each other.

The normal vector may be found as $\mathbf{N} = \mathbf{P_u} \times \mathbf{P_v}$. The tangent vector T may be taken as $\mathbf{P_u}$. The binormal vector $\mathbf{B}$ may be found as the cross product $\mathbf{B} = \mathbf{N} \times \mathbf{T}$.

The $\mathbf{TBN}$ basis is then

$$\mathbf{TBN} = \begin{pmatrix} T_x & B_x & N_x \\ T_y & B_y & N_y \\ T_z & B_z & N_z \end{pmatrix}$$

To transform vectors from object space to tangent space the inverse is needed

$$\mathbf{TBN}^{-1} = \mathbf{TBN}^{\mathbf{T}} = \begin{pmatrix} T_x & T_y & T_z \\ B_x & B_y & B_z \\ N_x & N_y & N_z \end{pmatrix}$$

To perform bump mapping using tangent space $\mathbf{TBN}^{\mathbf{T}}$ is used to transform the required object space vectors, e.g the light and viewer direction vectors into tangent space. The lighting calculation is then done as usual.