Topic 8
Particles and Pendula

References

(as per last lecture)

Alan Watt and Fabio Policarpo, 3D Games: Real-time Rendering and Software Technology, Addison-Wesley, 2001

Zill, D. G., A First Course in Differential Equations with Modelling Applications, Sixth Edition, Brooks/Cole Publishing Company, 1997.

David Baraff and Andrew Witkin, Physically Based Modeling: Principles and Practice, http://www.cs.c (originally appeared as SIGGRAPH 97 course notes)

Chris Hecker, Physics articles in Game Developer magazine, 1996-97, available at http://www.d6.com/users/

Numerical Integration Methods (review)

Last week, we learned of two intergration methods: Euler's Method, and Euler's Improved Method. We learned how Euler's Method used the current position (x_1, y_1) of a function $f(x, y)$ described in terms of its derivative, and the derivative at x_1 to estimate the new y_2.

We also learned how Euler's Improved Method uses an average of the gradient at x_1 and x_2 to improve on the estimated y_2. These are summarised below

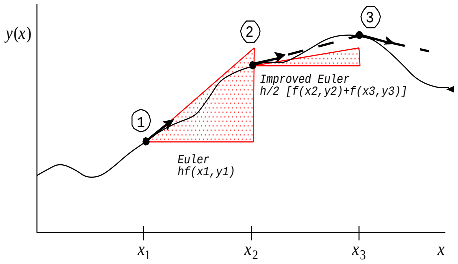


Figure 3: Euler's original and improved Methods.

Euler's method is biased by always using x_1. It can be shown that any errors will accumulate with little or no cancellation. In Euler's Improved Method, there is less bias and the errors are more likely to be either positive or negative.

The inadequacy of both methods lies in the attempt to use a linear extrapolation to estimate the new y. The Runge-Kutta method addresses this problem.

The Runge-Kutta Method

The Runge-Kutta method is a series approach where a polynomial is used to fit the curve to estimate the new y. Euler's Improved method is, in fact, a second order Runge-Kutta. The most popular form of Runge-Kutta method is the fourth order Runge-Kutta method.

The Runge-Kutta method evaluates the function at four different points and takes a weighted average of the gradient at each of those points.

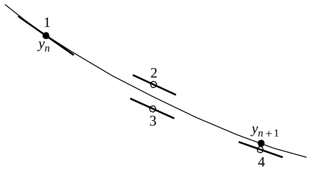


Figure 16.1.3. Fourth-order Runge-Kutta method. In each step the derivative is evaluated four times: once at the initial point, twice at trial midpoints, and once at a trial endpoint. From these derivatives the final function value (shown as a filled dot) is calculated.

Figure 4: Runge-Kutta Method.

Variable Step Size

All of the methods described so far use a fixed step size. There is no particular reason why a fixed step size needs to be used.

Comparing the Three Methods

Euler's Method.

$$
\begin{aligned}
c_1 &= hf(x, y), \\
y_{new} &= y + c_1
\end{aligned}
$$

Euler's Improved Method.

$$
\begin{aligned}
c_1 &= hf(x, y), \\
c_2 &= hf(x + c_1, y + h), \\
y_{new} &= y + \frac{1}{2}(c_1 + c_2)
\end{aligned}
$$

The Runge-Kutta Method.

$$
\begin{aligned}
c_1 &= hf(x, y), \\
c_2 &= hf(x + \frac{h}{2}, y + \frac{c_1}{2}), \\
c_3 &= hf(x + \frac{h}{2}, y + \frac{c_2}{2}), \\
c_4 &= hf(x + h, y + c_3), \\
y_{new} &= y + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)
\end{aligned}
$$

Newton's Laws (review)

The fundamental equations of mechanics are Newton's Laws of Motion:

1. *Law of Inertia.* A body will remain at rest or move with constant velocity unless acted upon by an external force.

2. The net force $\sum \mathbf{F}$ on a body with mass m is related to the acceleration $\mathbf{a}$ by

$$\sum \mathbf{F} = m\mathbf{a}$$

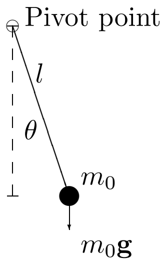
3. If body A exerts a force $\mathbf{F}_{BA}$ on body B, then B must exert a force $\mathbf{F}_{AB}$ on body A. The forces are equal in magnitude and opposite in direction:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

Pendulum Motion

The equations describing *pendulum motion* can also be derived from Newton's Laws of Motion.

Consider the pendulum. What are the forces acting on the ball?



The following points can be observed for a simple spring.

- The forces applied to the ball change direction over time.

- The string remains taut at all times, so the ball swings in a circular arc.

The angular velocity of the ball can be expressed as $\frac{d\theta}{dt}$, so the real velocity is $v = l\frac{d\theta}{dt}$ in the direction of increasing θ. Its acceleration is $a = l\frac{d^2\theta}{dt^2}$. The centripetal force towards the pivot is then $F_p = m_0 a$. The only other force is that due to gravity,

$$F_g = m_0\mathbf{g}(\cos\theta + \sin\theta) \tag{3}$$

of which one component $m_{\mathbf{g}}\cos\theta$ acts against the string. This only increases the string tension, and does not result in any motion. In our situation, the other component always acts in the direction of decreasing θ. These two forces thus work against each other, and we have

$$m_0 l\frac{d^2\theta}{dt^2} = -m_0\mathbf{g}\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{4}$$

This is another differential equation.

Equation 4 is not easily solvable analytically, but we intend to take small steps h. For small angles $\sin\theta \approx \theta$, so we can now say

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0, \quad \theta << 1 \tag{5}$$

Compare this with the spring equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{6}$$

So for very small angle perturbations, the behaviour of springs and of pendulums is similar. Both display what is called *Simple Harmonic Motion*, where a variable is proportional to its second derivative.

Large Swings of the Pendulum

The similarity between pendulums and springs is only true when we can approximate $\sin\theta$ with θ. So we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{7}$$

Since this is a second order D.E., we need to 'fix' the equations by setting two initial values. If we set $\hat{t} = tl/g$ so that we measure time in units of l/g, we get

$$\ddot{\theta} + sin\theta = 0$$

which is easier to handle. So we set

$$\theta = \theta_0, \quad \dot{\theta} = \phi, \quad \text{at } \hat{t} = 0$$

Projectile Motion

In projectile motion the only force is weight due to gravity

$$\sum \mathbf{F} = \mathbf{W} = m\mathbf{g}$$

If we define the y axis to be vertical and the positive direction to be away from the Earth's surface then the above equation can be written as a scalar equation

$$\sum \mathbf{F}_y = W = ma = -mg$$

which can be simplified to

$$a = -g$$

In projectile motion there is no force in the horizontal direction and thus the object's motion in the horizontal direction is governed by Newton's first law.

Defining the x axis to be in the direction of the horizontal motion of the object we get

$$\sum \mathbf{F}_x = 0$$

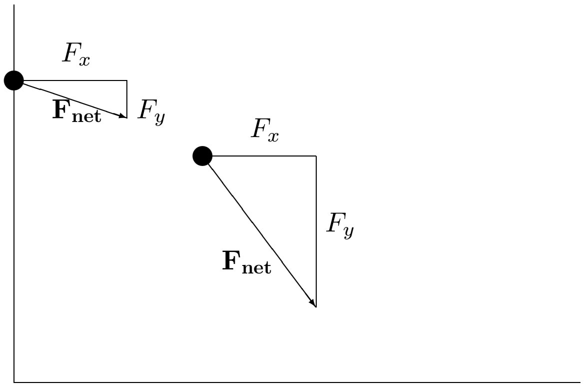


Figure 5: For a falling projectile, the horizontal component is constant, the vertical component is quadratic.

Projectile Motion: Differential Equation

The vertical component of projectile motion can be written as a differential equation:

$$a = \frac{d^2y}{dt^2} = y^{''} = \ddot{y} = -g$$

For projectile motion solving the ODE is straightforward: integrate twice and use initial conditions of at $t = 0$, $y = y_0$ and $v_y = v_{0_y}$.

$$\begin{aligned} y^{''} &= -g \\ \int y^{''} dt &= \int -g\, dt \\ y^{'} &= -gt + v_{0_y} \\ \int y^{'} dt &= \int (-gt + v_{0_y}) dt \\ y &= -1/2gt^2 + v_{0_y}t + y_0 \end{aligned}$$

Euler's Improved Method Applied to Second-Order IVPs

Forces give rise to accelerations, and accelerations are second derivatives of position with respect to time.

Thus methods for applying Euler's approach to second-order ODEs are important in physically based computer animation involving rigid body dynamics (including projectile motion).

To apply Euler's improved method to a second-order IVP it is turned into two (related) first-order IVPs to each of which Euler's improved method is separately applied.

Thus, to obtain an approximate solution to the second-order IVP

$$\begin{aligned} y^{''} &= f(x, y, y^{'}), \\ y(x_0) &= y_0, \\ y^{'}(x_0) &= y_1 \end{aligned}$$

we let $y^{'} = u$ and the problem then becomes

$$\begin{aligned} y^{'} &= u \\ u^{'} &= f(x, y, u) \\ y(x_0) &= y_0,\ u(x_0) = y_1 \end{aligned}$$

These ODEs are then solved numerically by applying Euler's improved method to each equation

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}(u_n + u_{n+1}) \\ u_{n+1} &= u_n + \frac{h}{2}(f(x_n, y_n, u_n) + f(x_{n+1}, y_{n+1}, u_n)) \end{aligned}$$

Summary

To quote Press *et al*, "Runge-Kutta is what you use when ... you don't know any better".

- The Runge-Kutta method almost always works
- Test for stability by re-running the integration at half the step size. If the numbers are very different, your simulation is unstable.
- Use variable step size when there are 'flat' regions in your function, interspersed with 'hilly' sections.