

David Halliday, Robert Resnick and Jearl Walker, Fundamentals of Physics, 6th Ed., John Wiley 2001

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## Topic 9 Multiple Spring Systems

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Multiple Spring Systems

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### Multiple Oscillation Modes

Recall that for a simple spring, the equation of motion was described as *simple harmonic* and related the position of a mass at the loose end of a spring to the force on that mass by the spring. i.e.

$$F = -kx(t), \text{ or}$$

$$\ddot{x}(t) + \frac{kx(t)}{m} = 0$$

and that a solution of the ODE was

$$x = A \cos \omega t + B \sin \omega t, \quad \omega^2 = \frac{k}{m}, \quad \ddot{x}(0) = 0$$

$$= C \cos(\omega t + D), \quad C = \sqrt{A^2 + B^2}, \quad D = \tan^{-1} \frac{k}{m}$$

Now imagine a mass attached with springs on both sides, with the other side of both springs attached to walls.

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### Doubly Anchored Spring in Series

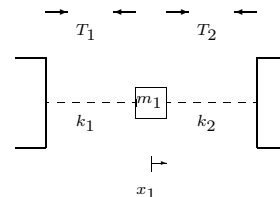


Figure 6: Doubly Anchored Spring in Series

A displacement of the mass by a distance  $x$  results in the first spring lengthening by a distance  $x$  (and pulling in the  $-x$  direction), while the second spring is compressed by a distance  $x$  (pushing in the  $-x$  direction). The equation of motion is thus

$$m_1 \ddot{x}(t) = -k_1 x(t) - k_2 x(t) = -(k_1 + k_2)x(t) \text{ so that}$$

$$\ddot{x}(t) = -\frac{k_1 + k_2}{m_1} x(t) \text{ and the frequency is} \quad (41)$$

$$\omega_{series} = \sqrt{\frac{k_1 + k_2}{m_1}} \quad (42)$$

The effective spring constant is thus the sum of the component spring constants,  $k_{eff} = k_1 + k_2$ .

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## Doubly Anchored Spring in Parallel

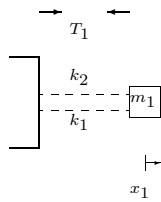


Figure 7: Doubly Anchored Springs in Parallel

Now when we attach two springs from the mass to the *same* wall, we find their forces add at all times. The effective spring constant is again the sum of the component spring constants, and  $\omega$  is also the same.

$$m_1 \ddot{x}(t) = k_1 x(t) + k_2 x(t) = (k_1 + k_2)x(t) \quad \text{so that}$$

$$\ddot{x}(t) = \frac{k_1 + k_2}{m_1} x(t) \quad \text{and} \quad (43)$$

$$\omega_{parallel} = \sqrt{\frac{k_1 + k_2}{m_1}} \quad (44)$$

But how are the above equations affected when there are two or more masses involved?

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## A Double Oscillation Mode Example

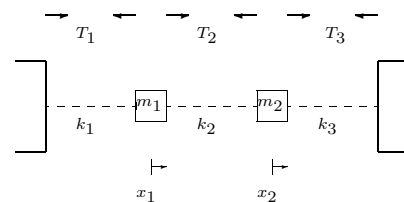


Figure 8: Double Mass and Spring

Now imagine two masses arranged as shown above. From your experience how would you expect them to behave when perturbed from their rest state?

The springs will have a natural tension

$$T_1 = kx_1, \quad T_2 = k(x_2 - x_1), \quad T_3 = -kx_3 \quad (45)$$

whenever the masses are perturbed (assuming identical springs,  $k = k_1 = k_2 = k_3$ ). So at time  $t$ , we have

$$F_{m_1} = T_2 - T_1 = m_1 \ddot{x}_1 = k(x_2 - 2x_1) \quad (46)$$

$$F_{m_2} = T_3 - T_2 = m_2 \ddot{x}_2 = k(x_1 - 2x_2) \quad (47)$$

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Now if we consider  $Y_1 = x_2 - 2x_1$  and  $Y_2 = x_1 - 2x_2$  as a different way of describing position, and substitute this into the above equation and  $m = m_1 = m_2$ , we have something that *looks* like SHM.

Since two ‘views’ are possible, one for each mass, this system is said to have two degrees of freedom, and in this case has up to two *Eigen-solutions*.

Since the system model looks like SHM, let’s try the corresponding solutions.

$$x_1 = A \cos \omega t, \quad \ddot{x}_1 = -A\omega^2 \cos \omega t$$

$$x_2 = B \cos \omega t, \quad \ddot{x}_2 = -B\omega^2 \cos \omega t$$

so that

$$\ddot{x}_1 = \frac{k}{m}(x_2 - 2x_1)$$

$$= \frac{k}{m}(B - 2A) \cos \omega t = -A\omega^2 \cos \omega t$$

$$\ddot{x}_2 = \frac{k}{m}(A - 2B) \cos \omega t = -B\omega^2 \cos \omega t$$

We can solve for  $\omega^2$

$$\omega^2 = \frac{-k(A - 2B)}{mB} = \frac{-k(B - 2A)}{mA}$$

$$= \frac{k}{m}, \quad \text{when } A = B, \quad \text{vibrating in-sync} \quad (48)$$

$$= \frac{3k}{m}, \quad \text{when } A = -B, \quad \text{vibrating anti-sync} \quad (49)$$

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Even with SHM, such modal behaviour can quickly become very complicated - indeed chaotic.

These two solutions are said to be *oscillation modes* of the system and such modal behaviour is common in situations where many elements are interacting. Note however, that the two modes occur given the initial conditions stated above. The more general case is really a mixture of the two modes such that

$$mA + nB = 0, \quad m, n \in \mathcal{I}$$

The above equations can also be applied to a double pendulum when the swing amplitudes are low ( $\sin \theta \approx \theta$ )

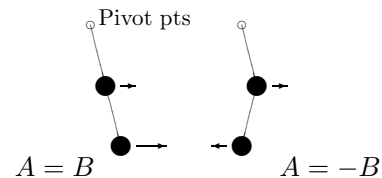


Figure 9: Two modes for Double Pendulums

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