Fractal Geometry and Chaos

References

[1] Benoit Mandelbrot, *The Fractal Geometry of Nature*, Freeman, 1982
[2] Michael Barnsley, *Fractals Everywhere*, Academic Press, 1993
[3] Kenneth Falconer, *Fractal Geometry*, Wiley, 1990
[4] Peitgen, Jurgens, Saupi, *Fractals for the Classroom parts I,II*, Springer, 1992
[5] Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* Springer, 1988
[6] Robert Devaney, *The Mandelbrot Set Explorer*, http://math.bu.edu/DYSYS/explorer/index.html
[7] N.C. Kenkel and D.J. Walker , *Fractals in the Biological Sciences*, http://www.umanitoba.ca/faculties/science/botany/labs/ecology/

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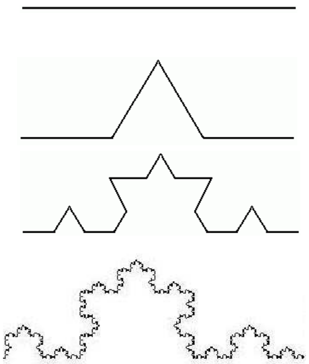
Iterated Function Systems

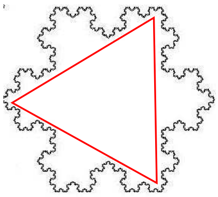
Recall in the last lecture, the Iterated Map Function.called the *Logistic* map. This is an example of an iterated function system. A more graphic example of this is in the generation of the **von Koch** Curve.

- 1st Generation
- 2nd Generation
- 3rd Generation
- Nth Generation



Each line in a generation is replaced iteratively by 4 lines

- Koch Snowflake



A von Koch snowflake is just 3 curve on a triangle substrate.

Observations of the von Koch Curve

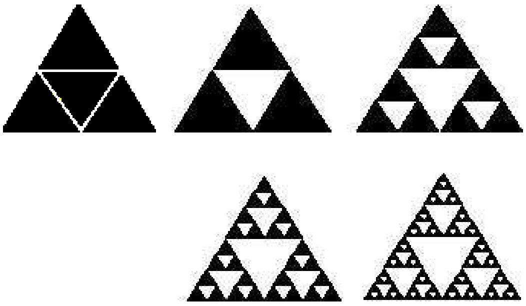
There are several things we can observe from the curve F:

- **F is self-similar.** It is clear that each generation is created from the preceding generation by repeated application of a rule.
- **F has 'fine-structure'.** As we zoom in to details of F we will find more of the same.
- F has a simple **recursive definition**.
- The geometry of F is **not easily described** in terms of traditional maths. It is not a path ('locus') travelled, nor the solution of an equation.
- The rule replaces an edge E with four edges whose combined length is longer than E. If the rule is applied infinitely, how long does that make E?
- The von Koch snowflake has **infinite perimeter, but finite area**.

These are some of the properties of a fractal geometry.

Sierpinsky Gasket

Here is a different object, to which we apply a rule recursively.



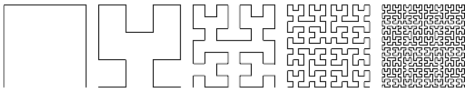
In each generation, the centre of each triangle is removed by joining the midpoints of the original triangle.

The Sierpinsky gasket is the set of all points in one plane that remain after an infinite number of generations.

Again, the fractal properties of self-similarity, fine-structure, and recursion are evident.

The Hilbert Curve

The Hilbert curve below is yet another fractal.



It can be constructed using a ‘turtle-graphics’ method of following directional instructions.

A set of string rewriting instructions can be used to ‘tell the turtle where to go’. The Hilbert curve is encoded with

- initial string y,
- string rewriting rules
  y → +XF-YFY-FX+,
  x → -YF+XFX+FY-,
  “turn angle is always 90° “

The first rule can be read as “turn 90° CW, substitute X, Forward one unit, turn 90° ACW, substitute Y, Forward, substitute Y, turn 90° ACW, Forward, substitute X, turn 90° CW. [5].

Similarly, the rule for the Koch curve is:

- Initial string f
- String rewriting rule f → F+F- -F+F
  “turn angle is always 60° “

Lindemayer systems (L-systems)

The above string rewriting rules were called a Lindenmayer system by Hilbert in 1891, and can be used to construct the Koch, Sierpinsky, Hilbert and many other figures which display the fractals properties.

In common with IFS, L-systems have at least two components – an initiator and a generator.

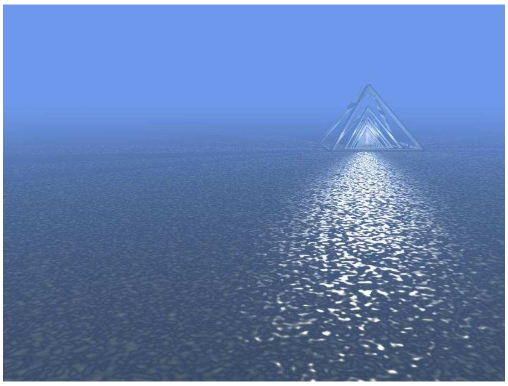
The **initiator** is the first construction from which we begin the fractal generation process (for the Koch snowflake, it is a triangle).

The **generator** is the process by which you proceed from one generation of the fractal to the next as was shown for the Koch curve.

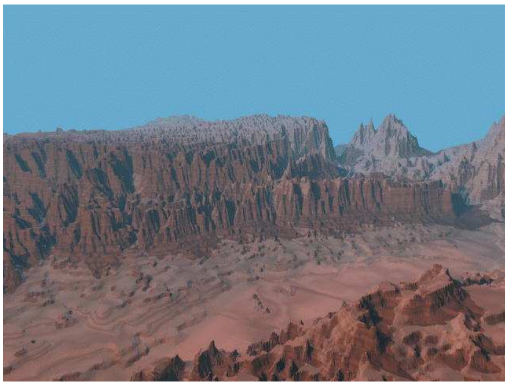
In general, the initiator defines the overall appearance of the form, whereas the generator defines the “amount” of fractal to apply to the initiator.

This “amount” determines the fractal properties (for example, the roughness) and is called the **Fractal Dimension**.

Examples of Graphics using L-Systems



Both these images built using Lparser and POVRay



http://www.xs4all.nl/~cvdmark/newgall.html

Leaves built using Lparser showing small mutations

http://www.xs4all.nl/~cvdmark/

Topological Dimension

An object is said to have a **topological dimension** roughly corresponding to the number of parameters needed to define it. So a rectangle has at least a height and width – however it was generated, it must always have a height and width to be called a rectangle.

A topological dimension satisfies the following properties

- **0-dimension**. A set of **unconnected** discrete points.
- **1-dimension**. A space where 0-dimensional objects are able to move continuously from one point to another. Ie. **A line**. The sequence of any points sharing the same 1-dimension can never change, just as numbers on a number line can never be reordered. A concept of **order** exists ("<", ">" have a unique meaning).
- **2-dimension**. A space where 1-dimensional objects are able to move orthogonally to their lengths. The concept of order is now no longer unique (how do you sort complex numbers?). The concept of **rotation** and **inside/outside** are now formed.
- **3-dimension**. A space where 2-dimensional objects are able to move orthogonally to their area-planes. The concept of rotation is no longer unique. The concept of the **tube** (inside *and* outside) is formed.

Some Other Users of Dimension

Statistics has a concept of **degree of freedom** in a problem or function, which relates to the the number of parameters in the problem which may be independently varied. One could plot these parameters on separate orthogonal axes. The result is called the n-dimensional parameter space of the problem.

Vector calculus defines Basis functions in terms of dimension. The familiar notation:

$$A = 3\underline{\mathbf{i}} + 4\underline{\mathbf{j}} + 5\underline{\mathbf{k}}$$

is really saying that we have a 3-dimensional vector. In vector algebra, the factors of the $\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$ unit vectors can only mix in certain specific ways.

Linear Algebra, being a generalisation of the vector, tensor and other calculi also decomposes quantities into basis vectors.

Even the humble **Complex Number** is two dimensional

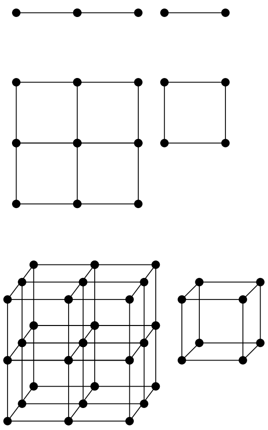
Scaling

Self-similar objects with can be described using a power-law:

From the figure it is apparent that there is a relationship between the number of pieces ***a***, needed to scale an object by a unit amount, and the scale by which we reduce each generation:

$$a = 1/s^d$$

where d = 1,2,3 for the line, square, and cube respectively.



Object	Number of pieces, **a**	Reduction scale, **s**
Square	**4, 4^2, 4^k**	**1/2, $1/2^2$, $1/2^k$**
Cube	**8^3, 8^2, 8^k**	**1/2, $1/2^2$, $1/2^k$**

The Hausdorff (or Fractal) Dimension

The power rule needs to rule generally because of the self-similar scaling that applies to fractal objects.

Now look back at the Koch and Sierpinsky curves and count the number of elements and the reduction facto for each generation. They are listed below.

Object	Number of pieces, **a**	Reduction scale, **s**	Fractal Dim, **d**
Square	**4, 4^2, 4^k**	**1/2, $1/2^2$, $1/2^k$**	**2**
Cube	**8^3, 8^2, 8^k**	**1/2, $1/2^2$, $1/2^k$**	**3**
Koch	**4, 4^2, 4^k**	**1/3, $1/3^2$, $1/3^k$**	**1.26**
Sierp..	**3, 3^2, 3^k**	**1/2, $1/2^2$, $1/2^k$**	**1.58**

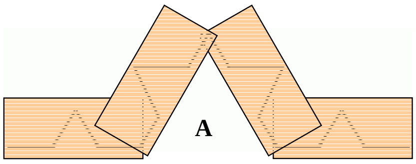
When we apply these to the formula

$$A = 1/s^d \quad \Rightarrow \quad d = \log(a) / \log(1/s)$$

we get a fractional quantity for the dimension as shown in the table.

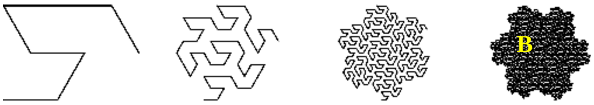
What does that actually mean?

Consider the Koch curve again:



For each unit, as we move over the unit, we pass 4 elements (lines in this generation) but have move a Euclidean distance of only 3 units. Now note that the width of the rectangles needed to contain the parts of the curve.

A very loose explanation of what happens is that the curve displays 1-dimensional geometry, but as you travel over the curve, you are also moving in an orthogonal dimension. The curve is not 2-dimensional because there are regions (A) that you cannot get to.



The **Peano-Gosper curve** above does not appear to have that problem. The unit length is 7, the scaling is 1/7, so D = log(7)/log(1/7) = 2.0. The dimension is 2 and the curve is said to be a **plane-** or **space-filling curve.** You *will* encounter B somewhere on the path.

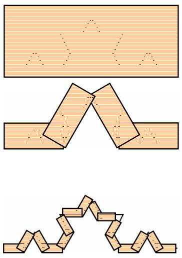
Measuring Fractal Dimension (FD) - Coastlines

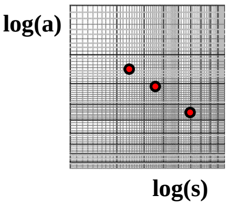
We can use Mandelbrot's example - determining the length of a country's coastline, which is not as simple as it first appears.

In fact, the answer depends on the length of the ruler you use for the measurements. A shorter ruler measures the smaller bays and inlets than a larger one, *so the estimated length continues to increase as the ruler length decreases.* This is the **coastline paradox**.

Plotting the length of the ruler versus the measured length of the coastline on a log-log plot gives a straight line, the slope of which is the fractal dimension of the coastline (and will be a number between 1 and 2).

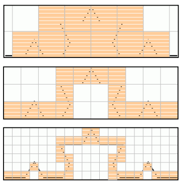
On the Koch curve, the gradient is 1.26.



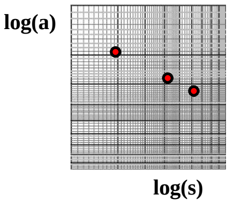


Measuring Fractal Dimension – Box Counting

Another way of measuring fractal dimension is to use different sized grids, and then to compare the resultant lengths by counting the boxes containing the curve.



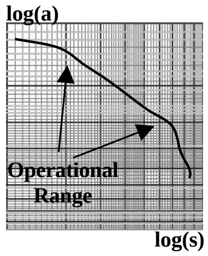
For example, the lengths of the three curves to the right are shown below



Number of pieces, **a**	Scale **s**
10	**3**
14	**2**
34	**1**

Again, if the logarithms of N and L are plotted, the gradient of the fitted line is the fractal dimension.

In real experiments, there will be factors such as data resolution and accuracy. The "operational range" is said to be the scale region where the FD line is straight.



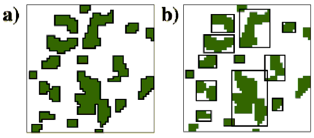
Typical Experimental Curve

Measuring Fractal Dimension – Area/Perimeter

Area-perimeter methods are generally used to estimate the fractal dimension of objects ('islands') coded as raster-based digitised images.

We can use the formula:

$$P \propto A^{D/2}$$



where A is the area of of eachobject, P the object perimeter and D the FD. Again, the *average* gradient of a log-log graph will reveal the FD. Note that this method can give very poor and biased results when the islands are small compared to the grid size, as coordinate quantisation and orientation errors will dominate.

There are many other methods of measuring FD. See [7] for more measures of FD.

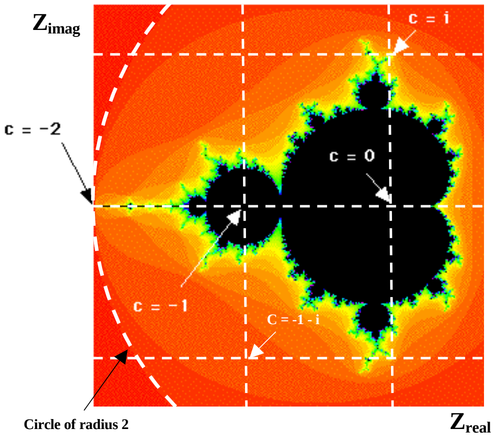
The Mandelbrot Set

The Mandelbrot set.[6] is the most famous of a class of iterated functions

In this case, the generator is

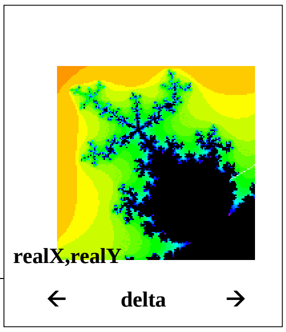
$$z \leftarrow z^2 + c$$

where z and c are complex quantities.



Basic Mandelbrot Algorithm

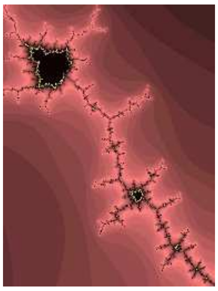
```
// (tX,tY) is current complex coordinate Z
// (cX,cY) is addition complex constant C


int i, j, index = 0;
double cX, tX, tY, tX1, tY1;
double cY = (double)realY;
double delta = (double)realDelta / frameWidth;

for (j = 0; j < frameWidth; j++) {
  cX = (double)realX;
  for (i = 0; i < frameHeight; i++) {
    tX = cX; tY = cY;

// Complex multiply (tX,tY)^2 + (cX,cY) ct iters
    for (ct = 0 ; ct < maxColour; ct++) {
      tX1 = (tX * tX);
      tY1 = (tY * tY);

// Bailout when we get outside the circle
      if (tX1 + tY1 > 4) break;
      tY = (tX * tY * 2) + cY;
      tX = tX1 - tY1 + cX;
    }
    plot(I,j,ct);
    index++;
    cX += delta;
  }
  cY += delta;
}
```



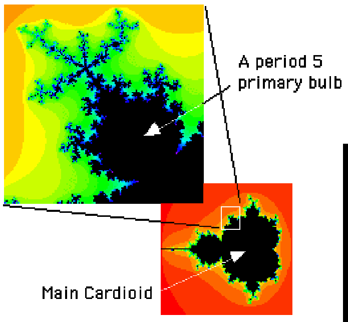
Exploring the Mandelbrot set

What happens to z as we proceed through the iterations depends on the position within the Mandelbrot set?

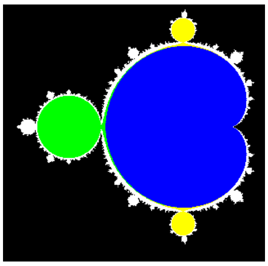
In the central bulb, z will cycle between a number of values. It is thus caught within the set. As we proceed toward the boundary, it takes longer and longer for this cycle to establish. On the boundary, the cycle never establishes, and eventually |Z| >= 2. When that happens the series no longer converges and we stop the loop, since z has *escaped*. The colour shown is related to the number of iterations required before |Z| >= 2 and is called the *escape time*.



As we wander through the set, we discover multiple copies of the original set as at right..



The different bulbs correspond to the periodicity of the cycles as the diagram below shows.



The **actual mandelbrot set** is really the boundary of the dark central region and has all the properties of a fractal. In particular, it is continuous, connected and non-differentiable.