

Halliday, D., Resnick, R. and Walker, J., Fundamentals of Physics, Fifth Edition, Wiley, 1997

Jeff Lander, *When Two Hearts Collide*, Game Developer, February, 1999

Topic 7 Collision Dynamics

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Collision Dynamics

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Collision Dynamics

“A *collision* is an isolated event in which two or more bodies (the colliding bodies) exert relatively strong forces on each other for a relatively short time.” (p215)

Collision dynamics is the dynamics (motion and forces over time) between colliding objects.

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Simplifying Assumptions

Collision dynamics are usually quite complicated.

They may be simplified by using the following assumptions:

- The objects are spherical particles.
- The particles can be modelled as *points*.
- The collisions are *completely elastic*.
- No forces apply between particles except for the instant they are colliding. In particular, no friction.
- The collisions are instantenous — the spheres are *hard*.

These assumptions are sometimes collectively referred to as billiard ball or hard sphere assumptions.

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The *center of mass* (x_{cm}, y_{cm}, z_{cm}) of a system of particles is a useful quantity in collision dynamics calculations.

The center of mass of a system of particles is a point defined as

$$\begin{aligned}x_{cm} &= \frac{1}{M} \sum m_i x_i \\y_{cm} &= \frac{1}{M} \sum m_i y_i \\z_{cm} &= \frac{1}{M} \sum m_i z_i\end{aligned}$$

where M is the sum of the masses of the particles

$$M = \sum m_i$$

Newton's laws of motion apply for a system of particles, as much as for a single particle.

Newton's second law for a system of particles is

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{cm}}$$

where \mathbf{F}_{ext} is the sum of *external* forces operating on the particles.

External forces are forces arising other than from the effect of one or more of the particles on another.

When $\mathbf{F}_{\text{ext}} = 0$ we get Newton's first law for a system of particles: no force, no change in velocity of the center of mass.

Conservation Principles

The study of mechanics — including dynamics — is often based around finding properties or quantities which are conserved.

Three quantities which are conserved under the hard sphere assumption are:

1. Momentum
2. Energy
3. Mass (non-relativistic)

Conservation of Momentum

The *momentum* \mathbf{p} of a single particle is defined as

$$\mathbf{p} = m\mathbf{v}$$

Note that this is a vector equation.

The momentum \mathbf{P} of a collection of particles is

$$\mathbf{P} = \sum m_i \mathbf{v}_i$$

Using Newton's second law it can be shown that

$$\sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{v}_{\text{cm}}}{dt} = M\mathbf{a}_{\text{cm}}$$

When there are no external forces $\sum \mathbf{F}_{\text{ext}} = 0$ Newton's first law gives us the principle of conservation of momentum

$$\mathbf{P} = m\mathbf{v}_{\text{cm}} = \text{constant}$$

Further, because mass is conserved (no particles leaving or joining the system) we get

$$\mathbf{v}_{\text{cm}} = \text{constant}$$

This is true even when the particles of the system are colliding and bouncing off each other — the velocity of the center of mass, and hence the momentum of the center of mass is constant.

Example

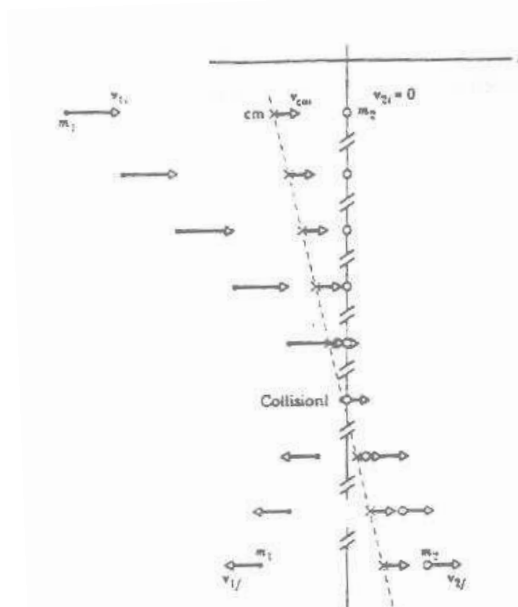


FIGURE 10-8 Some freeze-frames of two bodies undergoing an elastic collision. Body 2 is initially at rest, and $m_2 = 3m_1$. The velocity of the center of mass is also shown. Note that it is unaffected by the collision.

Example

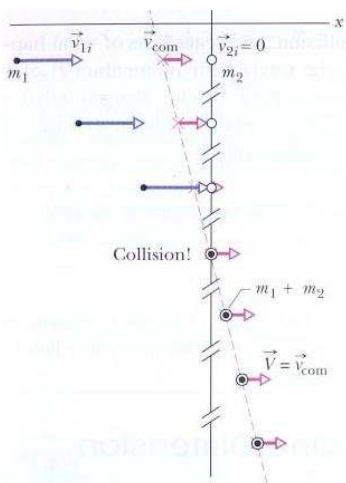


Fig. 10-10 Some freeze-frames of the two-body system in Fig. 10-9, which undergoes a completely inelastic collision. The system's center of mass is shown in each freeze-frame. The velocity \vec{v}_{com} of the center of mass is unaffected by the collision. Because the bodies stick together after the collision, their common velocity \vec{V} must be equal to \vec{v}_{com} .

Conservation of Energy

Another dynamical quantity which is conserved is energy.

Under the hard sphere conditions, we can assume the only form of energy of interest is *kinetic energy*.

The kinetic energy of a single particle is

$$K = \frac{1}{2}mv^2$$

Note that kinetic energy is a *scalar* quantity (not a vector quantity).

The total kinetic energy of a collection of particles is defined as the sum of the individual particles' kinetic energy

$$K = \frac{1}{2} \sum m_i v_i^2$$

It can be shown that when no external forces act on the system of particles that the total kinetic

energy of the system does not change — even when collisions occur.

Thus whilst (elastic) collisions might cause the kinetic energy of the individual colliding particles to change, the sum of their kinetic energy does not.

Furthermore, (elastic) collisions do not change the total kinetic energy.

Assume two particles with masses m_1 and m_2 are moving with velocities before colliding of v_{1i} and v_{2i} .

What are the velocities after collision?

We have two equations and two unknowns — solving them simultaneously will give the solution.

Conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad (34)$$

Conservation of (kinetic) energy:

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (35)$$

We want to solve for v_{1f} and v_{2f} .

Rewrite 34 as

$$m_1(v_{1i} - v_{1f}) = -m_2(v_{2i} - v_{2f}) \quad (36)$$

and 35 as

$$m_1(v_{1i}^2 - v_{1f}^2) = -m_2(v_{2i}^2 - v_{2f}^2) \quad (37)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) \quad (38)$$

Dividing 38 by 37 and rearranging gives

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \quad (39)$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \quad (40)$$

Equal Masses

What happens when the two masses are equal?

In 2D we have more equations and more unknowns.

Momentum — a vector quantity — is conserved in both the x and y directions.

Kinetic energy — a scalar quantity — is conserved.

This leads to five equations in ten variables: two masses m_1, m_2 ; and eight speeds $v_{1ix}, v_{1fx}, v_{1iy}, v_{1fy}, v_{2ix}, v_{2fx}, v_{2iy}, v_{2fy}$.

In the systems we are interested in there will be six knowns: the masses m_1, m_2 and the initial speeds $v_{1ix}, v_{1iy}, v_{2ix}, v_{2iy}$.

However, that leaves four unknowns and three equations.

The missing piece is to realise that when the particles collide the only forces which operate lie along the line between the centres. Thus this is the only direction in which the velocities change. The approach to solving the problem is to resolve the velocities into components along this direction and its perpendicular.

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Short-range Long-range Forces

Forces can generally be subdivided into two types:

- **Short-range Forces.** The forces act between two particles only when they are very close to each other.
- **Long-range Forces.** The forces acting between particles are still significant, even when the particles are not close to each other.

In our case, we can consider collision to be a short-range force, and the gravity or electrostatic forces between particles as a long-range forces.

For efficiency, when considering short-range forces only, for any particle, one needs only to consider the nearest neighbouring particle. For long-range force determination, *all* particles need to be considered.

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The conservation laws described previously work only for hard balls. In this case, the balls instantly change their momentum on contact. If the balls are soft, they may deform. This deformation uses up energy, which means that the energy after a collision is less than that before the collision. In the case where there are many balls, this may result in a gradual slow-down of all the velocities among the balls. Deformation can come in two varieties:

- **Irreversible deformation.** In this case, the ball is permanently deformed after the collision. The energy is absorbed by the deformation and is not re-released.
- **Reversible deformation.** The ball ‘bounces’. The energy required to deform the ball is stored as potential energy while the ball remains deformed, but is released again as kinetic energy after the collision. This usually takes the form of an additional force on the colliding ball.

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For each particle i , we thus write

$$F_i = F_{ext} + F_{SR} + F_{LR}$$

where ext is an external force, applied to all particles (normally 0).

For each particle we first calculate the LR forces, which require a calculation of interparticle distance. This requires a structure of the form:

```

for i = 1 to n
  T = 0
  for j = 1 to n
    if (j ~= i)
      T = T + calc_force(i,j)
    end j
  F(i) = T
end i

```

In the process, we keep track of which its nearest neighbour was. Then in determining the SR forces, we have only one particle to check for each particle.

The complexity of eq. can then be written

$$O = O(n) + O(n) + O(n^2)$$

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