Topic 12
Chaos

References

David Halliday, Robert Resnick and Jearl Walker, Fundamentals of Physics, 6th Ed., John Wiley 2001

David Acheson's Web site:
http://www.jesus.ox.ac.uk/ dacheson/programs

Exploring Chaos: Theory and Experiment
http://sunsite.anu.edu.au/education/chaos/index.html

Some demo web sites: http://bill.srnr.arizona.edu/ bill/der
http://wwwmaths.anu.edu.au/ briand/

The Double Oscillation Mode Revisited

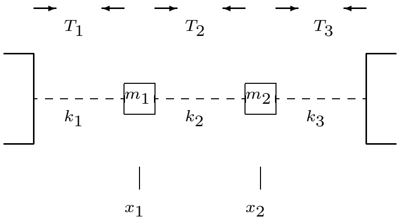


Figure 10: Double Spring Arrangements

Recall that two 'views' were possible, one for each mass, and that this system was said to have up to two *Eigen-solutions*, with resonant frequency ω^2 such that

$$\begin{aligned}\omega^2 &= \frac{-k(A-2B)}{mB} = \frac{-k(B-2A)}{mA} \\ &= \frac{k}{m}, \quad \text{when} \quad A = B, \quad \text{vibrating in-sync} \qquad (50) \\ &= \frac{3k}{m}, \quad \text{when} \quad A = -B, \quad \text{vibrating anti-sync} \qquad (51)\end{aligned}$$

The above equations can also be applied to a double pendulum when the swing amplitudes are low ($\sin\theta \approx \theta$)

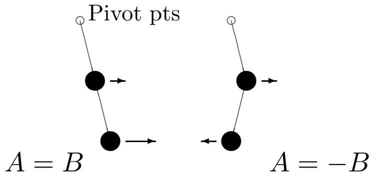


Figure 11: Two modes for Double Pendulums

The single and double pendula can also be *driven* with a forcing amplitude of a. The pivot point of the pendula is moved vertically with a time-varying height $h = a\cos\omega t$.

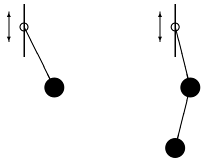


Figure 12: Two modes for Double Pendulums

The behaviour of both pendula can be chaotic under some circumstances, but generally there is a stable end state for both pendula. This end state may be

- **periodic**. For example, the single pendulum may swing around with a fixed period, or the double pendulum may form a dance which repeats itself.
- **convergent**. For example, the pendula rest vertically - up or down.

The process is chaotic because the ultimate end-state for any configuration is extremely sensitive to the initial conditions.

The Three Body Problem

When two planets are orbiting each other in stable orbits, their relative position is fairly straightforward. Henry Poincaré proved at the turn of the century that when a third body is added, the resultant motion cannot be analytically determined, and is often extremely complicated. He called it the N-body problem.

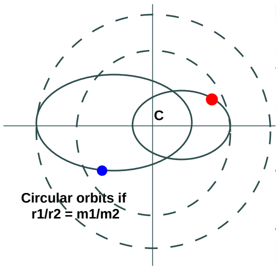


Figure 13: Two bodies

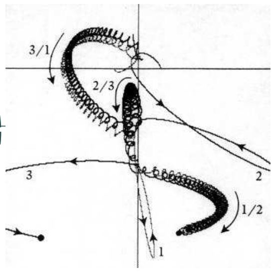


Figure 14: Three bodies

Chaos

From Newton's time, we were told that given good knowledge of essential parameters, we culd make accurate predictions of the future. For example, given the positions, masses and velocities of the planets, we could predict for long nito the future,

We now know that many dynamical systems in the universe are able to produce vastly different outcomes, given almost exactly identical initial conditions - the weather being an obvious example. This principle is sometimes called the 'Butterfly Effect.' In terms of weather forecasts, the 'Butterfly Effect' refers to the idea that whether or not a butterfly flaps its wings in a certain part of the world can make the difference in whether or not a storm arises one year later on the other side of the world. Sensitive dependence on initial conditions is one of the hallmarks of chaos.

Examples of Chaos - the Lorenz Model

Consider a system modelled by the following set of equations

$$\begin{aligned} \dot{x} &= 10(y-x) && (52) \\ \dot{y} &= rx - y - zx && (53) \\ \dot{z} &= -\frac{8}{3}z + xy && (54) \end{aligned}$$

This set of equations represents a fluid diffusion model in which r is the imposed difference temperature between the bottom of the fluid layer and the top.

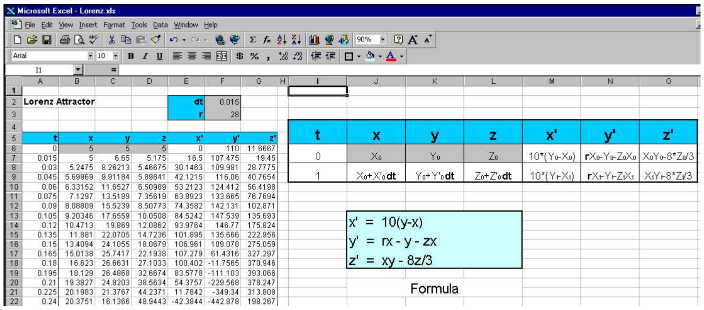
When $r < 1$, the system is stable at $x = y = z = 0$, whereas when $1 < r < 24.74$, the system has two stable regions centred on

$$x = y = \pm\sqrt{\frac{8}{3}(r-1)}, \quad z = r - 1$$

When r exceeds 24.74, the system behaves chaotically as shown below

Chaos using MSExcel

We can set up the Lorenz Model on MSExcel as shown below where we have one of two sheets visible. The r and dt variables are shown at top, and the initial x_0, y_0 and z_0 values are shown in gray in columns B to D, row 6. The second sheet is indentical except that $x_0 = 5.005$.



	A	B	C	D	E	F	G
1							
2	**Lorenz Attractor**				**dt**	0.015	
3					**r**	28	
4							
5	**t**	**x**	**y**	**z**	**x'**	**y'**	**z'**
6	0	5	5	5	0	110	11.6667
7	0.015	5	6.65	5.175	16.5	107.475	19.45
8	0.03	5.2475	8.26213	5.46675	30.1463	109.981	28.7775
9	0.045	5.69969	9.91184	5.89841	42.1215	116.06	40.7654
10	0.06	6.33152	11.6527	6.50989	53.2123	124.412	56.4198
11	0.075	7.1297	13.5189	7.35619	63.8923	133.665	76.7694
12	0.09	8.08809	15.5239	8.50773	74.3582	142.131	102.871
13	0.105	9.20346	17.6559	10.0508	84.5242	147.539	135.693
14	0.12	10.4713	19.869	12.0862	93.9764	146.77	175.824
15	0.135	11.881	22.0705	14.7236	101.895	135.666	222.956
16	0.15	13.4094	24.1055	18.0679	106.961	109.078	275.059
17	0.165	15.0138	25.7417	22.1938	107.279	61.4316	327.297
18	0.18	16.623	26.6631	27.1033	100.402	-11.7565	370.946
19	0.195	18.129	26.4868	32.6674	83.5778	-111.103	393.066
20	0.21	19.3827	24.8203	38.5634	54.3757	-229.568	378.247
21	0.225	20.1983	21.3767	44.2371	11.7842	-349.34	313.808
22	0.24	20.3751	16.1366	48.9443	-42.3844	-442.878	198.267

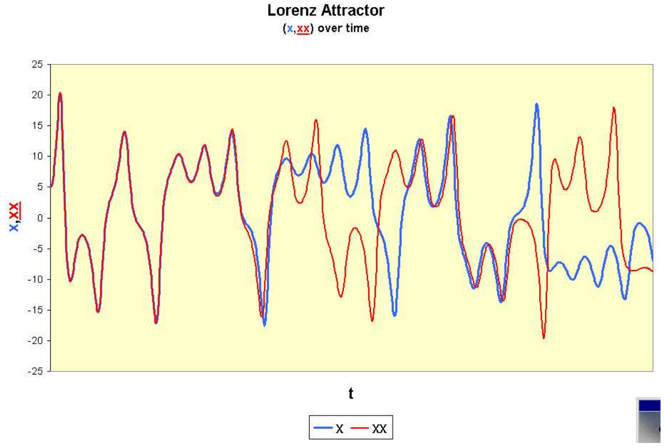
t	x	y	z	x'	y'	z'
0	X_0	Y_0	Z_0	$10*(Y_0-X_0)$	**r**$X_0-Y_0-Z_0X_0$	$X_0Y_0-8*Z_0/3$
1	$X_0+X'_0$**dt**	$Y_0+Y'_0$**dt**	$Z_0+Z'_0$**dt**	$10*(Y_1-X_1)$	**r**$X_1-Y_1-Z_1X_1$	$X_1Y_1-8*Z_1/3$

x' = 10(y-x)
y' = rx - y - zx
z' = xy - 8z/3

Formula

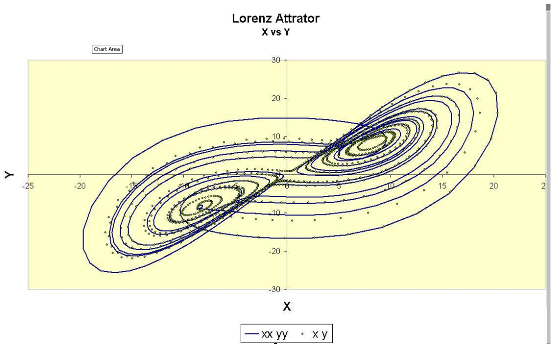
Time Evolution of the x Parameter

The plot marked x is the x parameter from the first sheet against time such that $x_0 = y_0 = z_0 = 5.0$, whereas the plot marked xx has $x_0 = 5.005$ as from the second sheet.



Phase Space Representation

A phase plot is often shown as a function x against its derivative, $\dot{x}$. The phase plot of the lorenz system shows its characteristic shape here. The line represents the phase of the second sheet, whereas for the dots it is the first sheet.



Two things to notice:

- The transition to randomness is sudden
- Extreme sensitivity to initial conditions

With a system so sensitive to initial condition, it is should be no surprise that it is sensitive to integration step size as well. Hence one key to determining the stability of any integration solution is the effect of a change of step size. You know you have a chaotic system if the effect of step size on the results is drastic.

So now we will look at the *process* of chaos in detail.

Routes to Chaos

The onset of chaos in a system can occur is various forms:

1. Period Doubling
2. Limit Cycles and Quasi-periodicity
3. Intermittency and Crisis

Routes to Chaos - Period Doubling

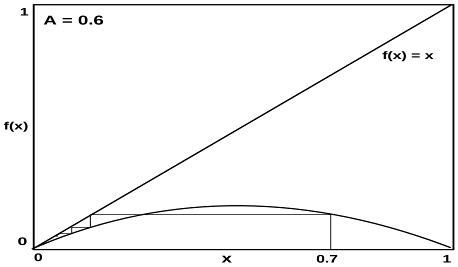
Consider the function

$$x_{n+1} = Ax_n(1 - x_n) = f_A(x_n)$$

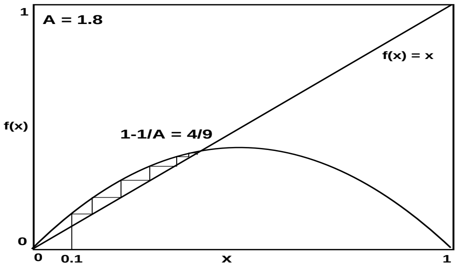
This function, called the logistic function (for historical reasons) might represent a population size of mayflies from year to year. This class of functions whose output maps back into the function on the next generation are called *iterated map functions*, and a group of such functions are called an *Iterated Function Systems*.

The sequence of x_n produced is called an *orbit* or *locus*. For some values of $x_n, f_A(x_n)$ is, or converges to, a fixed value. (either $f_A(x_n) = 0$ or $f_A(x_n) = 1 - \frac{1}{A}$).

For example, for $A = 0.6, x_0 = 0.7$, the locus of points (or orbit) converges to 0 as is shown below.



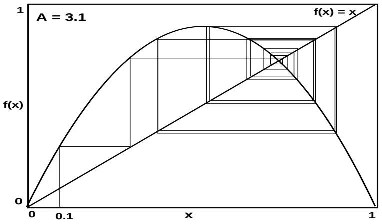
and for $A = 1.8, x_0 = 0.1$, then locus converges to $1 - \frac{1}{A}$



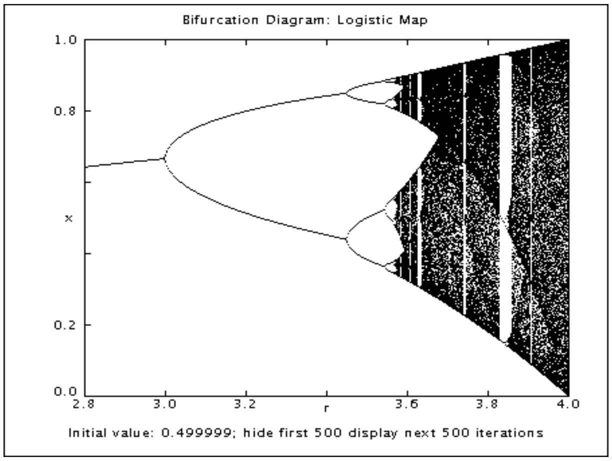
Routes to Chaos - Limit Cycles

Now notice what happens when $A = 3.1$! Although not obvious on the diagram below, the locus ends up in a regular cycle perpetually travelling around the convergent point without ever actually reaching it.

This behaviour is called a *limit cycle* and the intersection point is an *strange attractor*, with the region of x where all x_0 end up in a limit cycle about the intersection point called the *basin of attraction*.



If we were to pick a value of A, pick any x_0 between 0 and 1, and iterate 100 times to let the function approach an attractor, then plot the next 100 values of x, we get the *Bifurcation Graph* shown below.



The Feigenbaum Delta

There are many systems that show period-doubling behaviour en-route to chaos. The equation below (a *Sine Map*),

$$x_{n+1} = B\sin(\pi x_n)$$

shows exactly the same behaviour, but at different B values. We can now call the value of A where the first period doubling took place, $A_1 = 3$, and the point where period quadrupling occurred, $A_2 \approx 3.449$. If we did the same for B in the sine map, we find a remarkable property of these numbers.

Putting

$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$

Feigenbaum discovered that

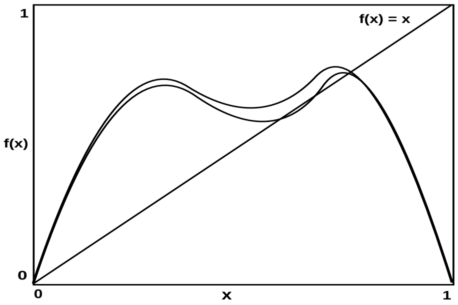
$$\lim_{n\to\infty} \delta_n = 4.66920161$$

which is now called the Feigenbaum delta. Subject to some constraints, this value is consistent for all iterated map functions.

Routes to Chaos - Intermittency/Crisis

The *Intermittency* route to chaos is characterised by dynamics with irregularly occurring bursts of chaotic behaviour, interspersed with intervals of apparently periodic behaviour. As some control parameter of the system is adjusted, the chaotic bursts become longer and longer, until the system is entirely chaotic.

A *crisis* is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly appears, disappears or changes size or position as a result of some control parameter being slowly adjusted. The figure below shows an example of this,



Chaotic Mixing - Stretch and Fold

As seen in the last slide, the function between x_n and $f_A(x_n)$ can cause a group of closely packed x_0 points to be spread over many iterations. An example of this can be seen in Acheson concerning the Rossler equations. In their usual form these are:

$$\begin{aligned}
\dot{x} &= -y - z \tag{55}\\
\dot{y} &= x + 0.2y \tag{56}\\
\dot{z} &= 0.2 + (x - c)z \tag{57}
\end{aligned}$$

The figure below shows $c = 5.7$. most of the time z is very small and x and y then evolve as

$$\begin{aligned}
\dot{x} &= -y \tag{58}\\
\dot{y} &= x + 0.2y \tag{59}\\
& \tag{60}
\end{aligned}$$

so that $\ddot{x} - 0.2\dot{x} + x = 0$.

Stretching causes paths which are close together to diverge over time, while folding causes the paths to cross each other in random ways.

An interesting feature of the Rossler equations is that the mixing is taking place entirely radially. In otehr words, the different paths are mixing their distances from the centre but they are all taking as long to travel around the circle.

In detail, trajectories on the central disk spiral outward from the centre. Eventually, they rise up in the z-direction and are then folded back down again. The disk itself is composed of an infinite number of layers, but these are so compressed that trajectories based at different initial conditions become effectively identical The result offers irreversibility and sensitivity to initial conditions - the hallmark of chaotic motion.