Parametric Equations
Polar, Cylindrical and Spherical Coordinates
References

- James Stewart, Calculus, 7th edition, 2011
Usually functions are described using *cartesian coordinates*, also referred to as *rectangular coordinates*, and *cartesian equations* in the form \( y = f(x) \).

Sometimes it is more convenient and concise to describe functions in a different way, where both \( x \) and \( y \) are described in terms of another, third variable, e.g. \( t, x = f(t) \) and \( y = f(t) \).

In this case \( t \) is called a parameter and \( x(t) \) and \( y(t) \) are *parametric equations*. In general \( t \) represents an arbitrary parameter, and not time (although it can be time).

Other commonly used parameters are \( s, u, v \).

If parameters are angles \( \theta \) (theta) \( \phi \) or \( \varphi \) (both phi), \( \psi \) (psi) or \( \alpha \) (alpha), \( \beta \) (beta), \( \gamma \) (gamma) might be used.
Parametric Equation Examples

- Example: line \( y = 2x + 3 \)
  
  \[
  x(t) = t \quad \text{(1)} \\
  y(t) = 2t + 3 \quad \text{(2)}
  \]

  or more succinctly
  
  \[
  x = t \quad \text{(3)} \\
  y = 2t + 3 \quad \text{(4)}
  \]

- Example: circle \( x^2 + y^2 = r^2 \)
  
  \[
  x = r \cos t \quad \text{(5)} \\
  y = r \sin t \quad \text{(6)}
  \]

  or using \( \theta \) as the parameter
  
  \[
  x = r \cos \theta \quad \text{(7)} \\
  y = r \sin \theta \quad \text{(8)}
  \]
The cartesian coordinate system specifies points in the plane by ordered pairs \((x, y)\) which are the distances from two perpendicular axes.

Similarly in 3D ordered triples \((x, y, z)\) are used.
Polar Coordinates

- The cartesian coordinate system specifies points in the plane by ordered pairs \((x, y)\) which are the distances from two perpendicular axes.

- **Polar coordinates** are another way to describe points in the plane, using \((r, \theta)\) ordered pairs.

- The origin \(O\) in polar coordinates is called the *pole* and a ray starting at \(O\) is called the *polar axis*. The distance \(r\) is the length \(OP\) and the angle \(\theta\) is measured counter clockwise from the polar axis.
Example: $(2, \pi/6)$
In polar coordinates functions are specified as \( r = f(\theta) \).

Example: \( r = a \) is a circle of radius \( a \) centred at the pole.

Example: \( \theta = \theta_0 \) is a straight line which makes an angle \( \theta_0 \) with the polar axis.

\( r = \cos \theta \) is a unit circle centred at (0.5, 0).
Polar Coordinates (cont)

- Polar coordinates can be converted to cartesian coordinates

\[ x = r \cos \theta \]  
\[ y = r \sin \theta \]  

- These are basically parametric equations, where \( x \) and \( y \) are given in terms of parameters \( r \) and \( \theta \).

- This assumes the pole is at the origin and the polar axis is aligned with the \( x \) axis.

- And rectangular coordinates can be converted to polar coordinates

\[ r = \sqrt{x^2 + y^2} \]  
\[ \theta = \tan^{-1} \frac{x}{y} \]
Cylindrical Coordinates

- Cylindrical polar coordinates, or just cylindrical coordinates, are 3D coordinates formed by adding to polar coordinates a third coordinate $z$ along the *longitudinal axis* to give $(\rho, \theta, z)$.

![Cylindrical Coordinates Diagram](image)

To convert from cylindrical to cartesian coordinates

\[ x = r \cos \theta \]  
\[ y = r \sin \theta \]  
\[ z = z \]  

\[ (13) \quad (14) \quad (15) \]
Spherical Coordinates

- Spherical polar coordinates, or just spherical coordinates, are also 3D coordinates given by ordered triples \((\rho, \theta, \phi)\).

- Again there is a pole or origin \(O\). The radial distance of \(OP\) is given by \(r\) (or \(\rho\)). There are two angles, the azimuthal angle \(\theta\) which measures the angle of the projection of \(OP\) onto the \(xy\) plane from the \(x\) axis and the polar angle or inclination angle \(\phi\) which measures the angle of \(OP\) from the \(z\) axis or zenith direction.
To convert from spherical coordinates to cartesian/rectangular coordinates

\[
x = \rho \sin \phi \cos \theta \\
y = \rho \sin \phi \sin \theta \\
z = \rho \cos \phi
\]  

(19) (20) (21)

Again these are essentially parametric equations. Instead of using \( \theta \) and \( \phi \), more general parameters e.g. \( u \) and \( v \) might be used, and instead of \( \rho \) use \( r \), which gives

\[
x = r \sin v \cos u \\
y = r \sin v \sin u \\
z = r \cos v
\]  

(22) (23) (24)
A torus can be similarly be specified using parametric equations with two angles and two radii. Using \( u \) and \( v \) for the angles, instead of \( \theta \) and \( \phi \) (which could equally be used) and \( R \) and \( r \) for the major and minor radii:

\[
\begin{align*}
  x &= (R + r \cos v) \cos u \\
  y &= (R + r \cos v) \sin u \\
  z &= r \sin v
\end{align*}
\]
A normal vector $\mathbf{N}$ on a 2D surface can be calculated as the cross product of two tangent vectors. Tangent vectors may be calculated using partial derivatives in $u$ and $v$:

$$P_u = \frac{\partial P}{\partial u} = ((R + r\cos v)\sin u, (R + r\cos v)\cos u, 0)$$

$$|P_u| = ((R + r\cos v)^2 \sin^2 u + (R + r\cos v)^2 \cos^2 u)^{1/2} = (R + r\cos v)$$

$$P_u/|P_u| = (-\sin u, \cos u, 0)$$

Similarly

$$P_v/|P_v| = (-\cos u \sin v, -\sin u \cos v, -\sin v)$$

And

$$\mathbf{N} = P_u/|P_u| \times P_v/|P_v| = (\cos u \cos v, \sin u \cos v, \sin v)$$